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9836. (Professor Lévy.)—Soient PA, PB, PC, PD les quatre normales issues d'un point quelconque P à une ellipse. Démontrer (1) que si la droite AB tourne autour d'un point fixe M, la corde CD enveloppe une parabole tangente aux deux axes ; (2) la réciproque ; et (3) trouver le lieu des foyers de ces paraboles lorsque le point M parcourt une droite donnée 48

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$$1-\frac{2n}{1.3}+\frac{2^2n(n-1)}{1.3.5}-\frac{2^3n(n-1)(n-2)}{1.3.5.7}+\ldots=\frac{1}{2n+1},$$

and that

$$1-\frac{n}{1.3}+\frac{n(n-1)}{1.3.5}-\frac{n(n-1)(n-2)}{1.3.5.7}+\ldots=\frac{1}{2n+1}\Sigma_0^n\left(\frac{C_x^n}{C_{2x}^{2n}}\right)\ldots\ldots 106$$

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$$(b+c)x^2+(c+a)y^2+(a+b)z^2 = ax^2+by^2+cz^2 = a+b+c,$$

prove that $(m\zeta-n\eta)^2+(n\xi-l\zeta)^2+(l\eta-m\xi)^2 = 2$, l, m, n being the direction-cosines of the tangent 102

15344. (R. W. D. Christie.)—Prove that any integer

$$N \equiv r^n . d_n + r^{n-1} . d_{n-1} + \ldots + s,$$

when (1) r = any radix whatever, (2) s = sum of the digits, (3) $d_n = n$-th differential divided by $n!$ *Ex.:*

(1) $12345 \equiv 9^4 . 1 + 9^3 . 6 + 9^2 . 15 + 9^1 . 20 + 15 = 12345,$

where $r = 10$, $s = 15$, $d_4 = 1$, $d_3 = 6$, $d_2 = 15$, $d_1 = 20$;

(2) $54321 \equiv 7^5 . 1 + 7^4 . 10 + 7^3 . 32 + 7^2 . 46 + 7^1 . 37 + 15 = 54321,$

where $r = 8$, $s = 15$, $d_5 = 1$, $d_4 = 10$, $d_3 = 32$, $d_2 = 46$, $d_1 = 37$ 49

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15391. (R. Chartres.) — Find the mean value of Z^n, where (1) Z = area of the triangle formed by joining two random points in a circle with the centre, (2) Z = volume of the tetrahedron formed by joining three random points in a sphere with the centre 49

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15409. (H. S. Vandiver.)—*A certain Aspect of Fermat's Theorem.*—If a is any given positive integer, then there is an infinity of composite integers M (each containing more than one distinct prime factor) such that $a^M \equiv a \pmod{M}$. For example, if $a = 2$, then $M = 11.31, 17.257, 23.89, \ldots$. If $a = 3$, then, $M = 7.13, 11.61, \ldots$ 26

15411. (Rev. T. Roach, M.A.)—Prove that the series

$$\frac{1}{1.2.4.5} + \frac{1}{3.4.6.7} + \frac{1}{5.6.8.9} + \ldots = \frac{1}{36} \ldots\ldots 34$$

15413. (Professor Sanjána, M.A.) — Prove the following three expressions equal, r being any positive integer :—

$$\frac{(r-1)!}{(1+x)^r}\left\{x^{r-1} - rx^{r-2} + \frac{r(r-1)}{1.2}x^{r-3} + \ldots + r\right\},$$

$$(-1)^{r-1}\frac{(r-1)!}{(1+x)^r}\left\{x^{r-1} - rx^{r-2}(1+x) + \frac{r(r-1)}{1.2}x^{r-3}(1+x)^2 - \ldots + (-1)^{r-1}r(1+x)^{r-1}\right\},$$

$$\frac{(r-1)!}{(1+x)^r}\left\{1 + (1+x) + (1+x)^2 + \ldots + (1+x)^{r-1}\right\} \ldots\ldots 40$$

15414. (Robt. W. D. Christie.)—In any right-angled triangle where the legs a, b differ by unity, and c, the hypothenuse, is an integer, prove $2 + 3c + 4a = x^2 + y^2$, and find an infinite number of integral values of a, b, c, x, and y 41

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$$p_n q_{n-1} - p_{n-1} q_n = \pm 1,$$

where p_n, q_n are convergents of the equation $P^2 - NQ^2 = \pm 1$. Prove the same property for the remainders $r_n q_{n-1} - r_{n-1} q_n = \pm 1$. *E.g.*, N = 73.—$p_n$ = 8, 9, 17, 94, 487, 581, 1068; q_n = 1, 1, 2, 11, 57, 68, 125; r_n = 0, 0, 1, 6, 31, 37, 68; then $68^2 - 37 \cdot 125 = -1$, $57 \cdot 37 - 68 \cdot 31 = +1$, ... 35

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15440. (H. Bateman, B.A.)—A sextic curve of the third class has cusps at the circular points and seven other cusps: these may be divided into sets of type A, B, C, such that the tangents at A, B, C meet in a point O. Prove that the Euler line of the triangle ABC passes through O and also through the triple focus 68

15441. (Professor Nanson.) — Three triangles Δ_1, Δ_2, Δ_3 are inscribed in a conic. If Δ_2, Δ_3 are co-polar with Δ_1, then so is the triangle formed by the joins of corresponding vertices of Δ_2, Δ_3 93

15443. (Professor Neuberg.)—Trouver le lieu des centres des hyperboles dont les asymptotes sont parallèles à deux droites données u, v et qui divisent harmoniquement deux droites données AB, CD..... 115

15445. (Lt.-Col. Allan Cunningham, R.E.) — *Analogue of Fermat's Theorem.*—Let $F_n = 2^{2^n} + 1$, a Fermat's number, and $P = F_a \cdot F_b \cdot F_c \ldots F_z$. Then $2^{P-1} \equiv +1 \pmod{P}$ and $2^{P(P-1)} \equiv +1 \pmod{P^2}$ for all values of P when $a < b < c < \ldots < z$, $a \not< 2$, and $z \not> 2^a - 1$ 55

15447. (Professor M. W. Crofton, F.R.S.) — Show that the mean value of the angle (θ) which the diameter of a semi-circle subtends at all points of its area is $M(\theta) = \frac{1}{4}\pi + 2/\pi$ 86

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15449. (James Blaikie, M.A.) — If u_1, u_2, u_3, ..., u_n be a series subject to the conditions (1) $u_1 = 1$, (2) $u_n = u_2 u_{n-1} + u_{n-2}$, prove the relations (3) $u_n = u_r u_{n-r+1} + u_{r-1} u_{n-r}$, (4) $u_{2r-1} = u_r^2 + u_{r-1}^2$,

(5) $u_{2r} = (u_{r+1}^2 - u_{r-1}^2)/u_2$, (6) $\Sigma_n u = (u_n + u_{n+1} - u_1)/u_2$.

This Question is in part a generalization of Question 15229. When $u_2 = 2$, the series becomes 1, 2, 5, 12, 29, ..., which is useful in connection with the solution of Question 15414 64

15450. (Professor Langhorne Orchard, M.A., B.Sc.) — Find the product of the sum and the difference of the series—

$1 + 6 + 21 + 52 + 105 + \ldots$ to n terms, $3 + 14 + 39 + 84 + 155 + \ldots$ to n terms 50, 93

15451. (A. Ensor.) — Find the form of n if the greatest coefficient in the expansion of $(1+x)^n$ is a multiple of $n(n-2)$ 70

15456. (D. Biddle.)—Three straight lines lie on a plane, no two of them parallel. Through a given point external to them draw (1) a straight line of which the intercept between two of the given lines shall be bisected by the third; (2) two straight lines forming with two of the given lines a quadrilateral, which is bisected by the third 103

15459. (C. E. Youngman, M.A.)—Three fixed circles S_1, S_2, S_3 touch two by two at A, B, C; and two variable circles V, V′, touching each other at P, are drawn so that V touches S_1 and S_2 and V′ touches S_1 and S_3. Show that the locus of P is the inverse of an ellipse, its trilinear equation in ABC being $8(\Sigma\beta\gamma \sin A)^2 = (Y + 3Z - X)(3Y + Z - X)$, where X, Y, Z stand for $\beta^2 + 2\beta\gamma \cos A + \gamma^2$, ... 54

15460. (S. C. Bagchi, B.A.)—A triangle moves so that its base is always parallel to the axis of a semi-cubical parabola, while its two sides and the median passing through their intersection are three tangents to that cubic. Prove that the locus of the vertex is the orthogonal plane projection of the same semi-cubical parabola, the axis being the axis of projection 70

15461. (Professor Neuberg.) — Couper un trièdre donné SABC par un plan qui rencontre les arêtes en des points A, B, C, tels que les projections de A sur SB et SC, de B sur SC et SA, de C sur SA et SB soient six points d'une même sphère 71

15462. (R. Chartres.) — If $1/x = \cdot\dot{a}bcd\ldots h\dot{k}$ in any scale, radix $= r$, then $p/x = \cdot\dot{k}abcd\ldots\dot{h}$. Find p, and show that there is an infinite number of scales where $1/x$ gives the same number of digits as above, and where p/x can be obtained by putting the last digit first. Give also the general value for the radix 46

15463. (Robt. W. D. Christie.) — Solve $X^2 - 44537Y = -1$, and generalize your solution 84

15464. (C. M. Ross.)—Prove that the continued fraction

$$\frac{a}{(a+1)-}\ \frac{b}{(b+1)-}\ \frac{c}{(c+1)-}\ \ldots \text{ ad inf.}$$

has the value unity if a, b, c, ... are positive integers 69

15467. (Professor M. W. Crofton, F.R.S.) — A floor is ruled with parallel lines 1 foot asunder. If an equilateral triangle whose side is 1 foot be thrown on the floor, show that the chance of it falling on one of the lines is $3/\pi$ 56

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15468. (R. Tucker, M.A.)—ABC is the circum-circle of a triangle. Ap, Ap' are lines drawn from A to p, p' on the circumference, each equal to the perpendicular from A on BC. Bq, Bq'; Cr, Cr' are like lines from B, C. Prove that pp', qq', rr' are the respective antiparallels to the angles A, B, C, and by their intersections form a triangle abc similar to the orthocentric triangle of ABC. Prove further that Aa, Bb, Cc intersect in K, the symmedian point of ABC 113

15469. (Lt.-Col. Allan Cunningham, R.E.)—Find a series of pairs of right-angled triangles whose sides (a, b, c; a', b', c) are integers, and with a common hypothenuse (c), and such that $c-b = b-b' = 1$..... 46

15470. (Salutation.) — The sines of two angles are expressible in proper fractions the denominators of which differ by unity; and their cosines are inversely proportional to the respective denominators. The latter being given, find the several numerators 79

15472. (Professor Neuberg.) — On marque sur les côtés BC, CA, AB d'un triangle ABC les points A', B', C' tels que $BA'/A'C = CB'/B'A = AC'/C'B = K$. On sait que les droites AA', BB', CC' sont équipollentes aux côtés d'un triangle A''B''C''. L'aire de ce triangle est minimum lorsque $K = 1$..... 94

15473. (C. E. Youngman, M.A.)—From A, B, C lines AEF', BFD', CDE' are drawn making (all in the same sense) an angle θ with BC, CA, AB respectively; and, again, AE'F, BF'D, CD'E doing the same, but in the other sense. Prove that ABC, DEF, D'E'F' are in perspective two by two, having the same axis, whose direction is constant, and that two of the centres P, P' lie on the Kiepert hyperbola and the third Q on the isogonal transformation of the Euler line OH. When $\theta = 60°$, Q is the "twin" of O 115

15474. (*Communicated by* Arthur Anstey, B.A., L.C.P.)—ABC is an isosceles triangle, base angles being 72°; $\angle EBC = 25°$, $\angle DCB = 45°$; DE is joined and produced to meet the base produced in F. Find the angle EFC 61

15478. (Professor Nanson.)—If r, θ are the focal polar co-ordinates of the foot of a normal from r', θ' to a conic, prove that $r_1r_2r_3r_4 = b^2e^{-2}r'^2$, $\theta_1+\theta_2+\theta_3+\theta_4 = 2\theta'+(2n+1)\pi$ 94

15481. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely $N = 96^{18}+1$ 62

15482. (James Blaikie, M.A.) — Find two numbers of eleven digits such that in each case the square of the number is terminated by the eleven digits of the number; and show that the sum of these two numbers is necessarily $10^{11}+1$ 71

15484. (Lt.-Col. J. Hickman.)—Smith takes a number in a small sweepstakes of 100 chances at 2s. 6d. each on a race in which 12 horses are entered. After drawing he forgets the name of the horse he drew, remembering only that it began with the letter *N*. When the race has been run he learns that the name of the winner begins with an *N*. The only further light he has as to the probability of his having won is that he believes $\frac{1}{20}$ of all racers bear names beginning with *N*. Proceeding on that assumption, find the precise mathematical value of his chance if the winner is to take $\frac{1}{3}$ of the stakes 91

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15487. (A. M. Nesbitt, M.A.)—Resolve into factors

$$\begin{vmatrix} a & . & . & l & . & . \\ . & b & . & . & m & . \\ . & . & c & . & . & n \\ g & . & . & d & . & . \\ . & h & . & . & e & . \\ . & . & k & . & . & f \end{vmatrix}$$

..... 86

15490. (R. Tucker, M.A.)—Find the envelope of a parabola which cuts a given parabola in four points three of which are given co-normal points 105

15491. (L. Isserlis.) — On a circle centre O are taken points p, q, r such that the arcs pq, qr are equal. Lines pP, qQ, rR are drawn parallel to a common tangent of the circle and a concentric conic meeting the latter in P, Q, R. Show that the tangent at Q to the conic is parallel to PR 112

15492. (W. J. Greenstreet, M.A.)—Through the feet of the normals from a point P to a rectangular hyperbola passes another rectangular hyperbola such that its normals at these four points are concurrent. Find the point of concurrence and the locus of P if the two curves are equal..... 114

15493. (*Communicated by* C. M. Ross.) — Two straight lines move between two rectangular axes in such a way that their ends are always concyclic. Find the locus of the centre of the circle through the ends of these lines 67

15503. (Robt. W. D. Christie.) — Prove that the so-called Pellian equation may be reversed by the *dual principle* as follows:—Since

$$(4a^3+6a^2+6a+2)^2-(4a^2+4a+5)(2a^2+2a+1)^2=-1$$

for all values of the letter, then

$$(2a^3+6a^2+6a+4)^2-(5a^2+4a+4)(a^2+2a+2)^2=-a^6.$$

Give a few other examples, and show how to introduce the prime roots 119

15508. (R. F. Davis, M.A.) (Suggested by Mr. C. E. Youngman's Question 15435.)—Three circles touch each other externally and touch externally the circum-circle of a given triangle ABC at A, B, C respectively. Show that their radii are R/(cot B + cot C − 1), ... 111

15513. (J. Macleod, H.M.I.S., I.S.O.) — Using the usual notation for the ellipse, if at S a perpendicular SD to the major axis is raised equal to SP and in this perpendicular a point E is taken such that SE = AN, prove that $\triangle$TDE = $\triangle$ADS. In what case does the tangent PT coincide with TD? 108

15520. (Robt. W. D. Christie.)—Solve generally

$$A^3+B^3=(a^3+b^3)(c^3+d^3)$$

in integers 115

15523. (A. M. Nesbitt, M.A.) — A man, who has m hats of his own in his hall, is visited by n friends, each wearing a hat. They leave their hats with those of their host. When they are going away they are unfortunately not in a condition to distinguish between one hat and another. Find the chance that no guest takes away his own hat ... 113

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15524. (R. Chartres.) (Suggested by Question 15447.) — Find the mean value of the angle subtended by BC at all points of the area BAC, (1) when BAC is a segment of a circle and $A = \pi/n$, (2) when BAC is any triangle 96

15532. (*Communicated by* R. F. Whitehead, B.A.) — The ends of a room are 12 ft. square and its length is 30 ft. Show that the shortest path a spider situated 3 ft. from the ceiling on the mid-line at one end of the room can take towards a fly at the diametrically opposite point on the other end wall is about 41 ft. 9½ in. 111

15538. (W. H. Blythe, M.A.)—There are N bullets in a bag to be divided among n people. The first takes what he considers to be his share, finding one remainder, which he lays aside. The second, not knowing the first has removed any of the bullets, does exactly the same thing, and each of the people in turn repeats the process. Finally, the remaining bullets are divided into n equal portions, with one remainder, each person taking his share. Show that N is of the form $An^p - n + 1$, where A is any positive integer and p is not less than $n + 1$. Find an expression for the share of each person..... 109

15540. (Robt. W. D. Christie.) — Solve the equation in positive integers $[\frac{1}{2}(p+e-f)][\frac{1}{2}(p+f-e)] - p[\frac{1}{2}(p-e-f)] = 1$, when p is any prime whatever of form $4m+1$. Will it be true also for n? 118

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

15353. (H. B. Reynolds.)—In a house are two cisterns respectively 3 ft. cube and 4 ft. cube. It is required to replace these by two new cube cisterns, one less than 3 ft. cube, the other more than 4 ft. cube, so that the two new ones may together contain the same as the two original ones, *i.e.*, 91 cubic feet. The answer must be exact in vulgar fractions —not an approximation in decimals.

Solution by D. Biddle.

Let $(a-u)^3+(b+v)^3 = a^3+b^3$, where $a<b$, and, of necessity, $u>v$. The answer being required in exact vulgar fractions, let z be the L.C.M. of the denominators of u and v, and let $u = x/z$, $v = y/z$. Then we obtain

$$(az-x)^3+(bz+y)^3 = (a^3+b^3)\,z^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

whence $x^3-y^3+3bz\,(x^2-y^2)+3b^2z^2\,(x-y) = 3\,(a+b)\,xz\,\{x+(b-a)\,z\}$...(2)

and

$$\{(a+b)\,z-(x-y)\}\{(az-x)^2-(az-x)(bz+y)+(bz+y)^2\} = (a^3+b^3)\,z^3 \ldots(3).$$

From (2), we have $x^3-y^3 \equiv 0 \pmod{3z}$.....(4)

and $3\,(a+b)\,xz\,\{x+(b-a)\,z\} \equiv 0 \pmod{x-y}$ .....(5).

From (3), we have $(a^3+b^3)\,z^3 \equiv 0 \{\mathrm{mod}\,(a+b)\,z-(x-y)\}$ .....(6),

whence $(a^2-ab+b^2)\,z^2\,(x-y) \equiv 0 \{\mathrm{mod}\,(a+b)\,z-(x-y)\}$.....(7).

Taking (4), let $x-y = n$. Then x^3-y^3 becomes $3nx^2-3n^2x+n^3$, and, since 3 divides the whole, it divides n. Let $n = 3n_1$; then, from (3) and (7), we have

$$x^3-3n_1x+3n_1^2+\{(b-a)\,x-(2b-a)\,n_1\}\,z = \frac{(a^2-ab+b^2)\,n_1z^2}{(a+b)\,z-3n_1} \quad \ldots(8).$$

From (5), we now have $(a+b)\,xz\,\{x+(b-a)\,z\} \equiv 0 \pmod{n_1}$.....(9).

From (8) and (9), $zx \equiv 0 \pmod{n_1}$ .....(10),

whence also $zy \equiv 0 \pmod{n_1}$ .....(11).

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In the present case n_1 is a factor of x and of y, but this does not appear to be necessary. It must not be supposed, moreover, that n_1 enters into the composition of z as well as x and y, for, if such were the case, u and v would not be reduced to their lowest terms.

Now, if we regard the cubes as varying gradually whilst their sum remains constant, the differential calculus teaches us that the infinitesimal element first taken from the edge of the smaller cube will bear to that first added to the edge of the larger the inverse ratio $3b^2 : 3a^2$. Therefore we are not surprised to find that the common denominator of the required (exact) fractions is closely related to $3(b^2-a^2)$, and in the case before us this is the actual value of z; also, since $b-a=1$, z here $= 3(a+b)$. But it is possible that in some cases z may be a multiple, or on the other hand a factor, of $3(b^2-a^2)$. Or it may occur, as indicated by (8), that z divides $x^2-3n_1x+3n_1^2$, although in the present case they only have a common factor. But in all cases, since $x^3-y^3 = 9n_1(x^2-3n_1x+3n_1^2)$, we have, by (4),

$$3n_1(x^2-3n_1x+3n_1^2) \equiv 0 \pmod{z} \quad \ldots\ldots(12),$$

and, by (8),

$$3(a^2-ab+b^2)n_1^2z \equiv 0 \left\{\mathrm{mod}\,(a+b)z-3n_1\right\} \quad \ldots\ldots(13),$$

which should be taken in conjunction with (7). Let

$$(a^2-ab+b^2)n_1z^2 = \lambda\left\{(a+b)z-3n_1\right\} \quad \ldots\ldots(14)$$

and

$$3(a^2-ab+b^2)n_1^2z = \mu\left\{(a+b)z-3n_1\right\} \quad \ldots\ldots(15).$$

Then $\lambda : \mu = z : 3n_1$, and we obtain

$$(a+b)\lambda-\mu = (a^2-ab+b^2)n_1z \quad \ldots\ldots(16),$$

and it is clear, from (14) and (15), that

$$3(a^2-ab+b^2)n_1z \equiv 0 \left\{\mathrm{mod}\,(a+b)z-3n_1\right\} \quad \ldots\ldots(17),$$

which is an advance upon both (7) and (13). Let

$$3(a^2-ab+b^2)n_1z = \rho\left\{(a+b)z-3n_1\right\} \quad \ldots\ldots(18);$$

then $n_1\rho = \mu$, whilst $z\rho = 3\lambda$, whence, by (16),

$$\rho^2 = 3(a^2-ab+b^2)\lambda\mu/\left\{(a+b)\lambda-\mu\right\} \quad \ldots\ldots(19).$$

In the present case $\rho = \frac{1}{3}n_1z$, and, since $z = 3(a+b)$, we have

$$n_1 = (a+b)^2-3(a^2-ab+b^2).$$

But, in (12), take

$$3n_1(x^2-3n_1x+3n_1^2) = mz \quad \ldots\ldots(20);$$

then

$$x = \left\{(12mn_1z-27n_1^4)^{\frac{1}{2}}+9n_1^2\right\}/(6n_1) \quad \ldots\ldots(21),$$

whilst

$$y = \left\{(12mn_1z-27n_1^4)^{\frac{1}{2}}-9n_1^2\right\}/(6n_1) \quad \ldots\ldots(22).$$

In the present case $m = n_1^2$, giving $x = 40$, $y = 10$, with $z = 21$, n_1 being 10. Hence the required internal measurements of the new cisterns are $1\frac{2}{21}$ ft. and $4\frac{10}{21}$ ft. respectively.

15409. (H. S. Vandiver.)—*A certain Aspect of Fermat's Theorem.*—If a is any given positive integer, then there is an infinity of composite integers M (each containing more than one distinct prime factor) such that $a^M \equiv a \pmod{M}$. For example, if $a=2$, then $M = 11.31, 17.257, 23.89, \ldots$. If $a = 3$, then, $M = 7.13, 11.61, \ldots$.

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Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Raise the given a to any power ξ, and find two proper (unequal) prime factors (say, p, q) of $(a^\xi - 1)$ so that $a^\xi \equiv 1 \bmod p$ and $\bmod q$, and therefore also $\bmod pq$, where $p = m\xi + 1$, $q = n\xi + 1$. Here

$$pq = (mn\xi + m + n)\xi + 1.$$

But $(a^\xi)^{mn+m+n} \equiv 1 \pmod{pq}$; therefore

$$a^{pq-1} \equiv 1 \pmod{pq} \quad \text{and} \quad a^{pq} \equiv a \pmod{pq}.$$

15365. (Professor SANJÁNA, M.A.)—In a parabola PQ is the normal chord at P, and AN the abscissa of P. If P_1P and P_2P are also normal chords, prove that $P_1P.P_2P = PQ.AN$.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by* R. TUCKER, M.A.; (III.) *by* H. J. WOODALL, A.R.C.S.

(I.) Let the tangents at P_1, P_2 intersect in T. Then it may be shown that ST is collinear with, and equal to, SY, the perpendicular from S on the tangent at P. Hence $SP_1.SP_2 = ST^2 = SY^2 = SA.SP$. Moreover, $AN_1.AN_2 = 4SA^2$. The normal chord PQ at $P = 4SP^{\frac{1}{2}}.SA^{\frac{1}{2}}/AN$. Hence the relation as stated.

(II.) Let P, P_1, P_2 be the three points m, m_1, m_2; then

$$m + m_1 + m_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Then PQ, being a normal chord, is the join of m, $-(m^2+2)/m$; therefore

$$\text{its length} = 4a\sqrt{(m^2+1)^3}/m^2 \quad \ldots\ldots\ldots\ldots \quad (2).$$

$$\begin{aligned} PP_1.PP_2 &= \sqrt{\left[(m^2-m_1^2)^2 + 4(m-m_1)^2\right]}\sqrt{\left[(m^2-m_2^2)^2 + 4(m-m_2)^2\right]}\,a^2 \\ &= (m-m_1)(m-m_2)\sqrt{\left[(m_1^2+4)(m_2^2+4)\right]}\,a^2 \text{ by (1)} \\ &= 2(m^2+1)\sqrt{\left[m_1^2m_2^2 + 4(m_1^2+m_2^2) + 16\right]}\,a^2 \\ &= 4(m^2+1)^{\frac{3}{2}}a^2; \end{aligned}$$

because $m_1m_2 = 2$, since the conormal point is P (see *Co-normal Points on a Parabola*, § 15, $\Upsilon = 2am$, and also $= amm_1m_2$); therefore

$$P_1P.P_2P = AN.PQ.$$

(III.) If P be $(am^2, -2am)$, PQ will be $y = mx - 2am - am^3$, giving $Q\left[a(m^2+2)^2/m^2,\ 2a(m^2+2)/m\right]$; therefore

$$AN.PQ = 4a^2(1+m^2)^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

$y = Mx - 2aM - aM^3$ will pass through $(am^2, -2am)$ if

$$-2am = 2a(M^2+2)/M;$$

whence

$$M^2 - mM + 2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

If M_1, M_2 are the roots of this quadratic, corresponding to P_1P, P_2P respectively, then

$$\begin{aligned} P_1P.P_2P &= 4a^2(1+M_1^2)^{\frac{3}{2}}\,4a(1+M_2^2)^{\frac{3}{2}}/M_1^2M_2^2 \\ &= 16a^2(1+M_1^2+M_2^2+M_1^2M_2^2)^{\frac{3}{2}}/M_1^2M_2^2 = 4a^2(1+m^2)^{\frac{3}{2}} \ldots (3) \end{aligned}$$

therefore $AN.PQ = P_1P.P_2P$.

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12880. (Professor MATTHEWS, M.A.)—Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{16}+\ldots$ ad inf. $=\tan^{-1}(-\tanh\pi/\sqrt{2}\cot\pi/\sqrt{2})$, when all the $(\tan^{-1})$'s are to be taken between 0 and $\frac{1}{2}\pi$; and verify this result numerically, so as to show the equality of the two expressions within 1 degree of angle.

Solutions (I.) *by* Professor SANJÂNA, M.A.; (II.) *by* F. P. ANDERSEN.

(I.) In the identity $\sin\alpha=\alpha\Pi_1^\infty(1-\alpha^2/n^2\pi^2)$, suppose α to be the complex number $x+\iota y$. Taking logarithms, we get

$$\log(\sin x\cosh y+\iota\cos x\sinh y)=\log(x+\iota y)+\Sigma_1^\infty\log\left(1-\frac{x^2-y^2}{n^2\pi^2}-\iota\frac{2xy}{n^2\pi^2}\right).$$

Taking principal values only, we have the imaginary part on the left $=\iota\tan^{-1}(\tanh y\cot x)$; that on the right

$$=\iota\tan^{-1}\frac{y}{x}+\iota\Sigma_1^\infty\tan^{-1}\frac{-2xy}{n^2\pi^2-x^2+y^2}.$$

Put $x=y=\pi/\sqrt{2}$, and cancel ι;

$$\tan^{-1}(\tanh\pi/\sqrt{2}\cot\pi/\sqrt{2})=\tan^{-1}1+\Sigma_1^\infty\tan^{-1}(-1/n^2)$$
$$=-\tan^{-1}\tfrac{1}{4}-\tan^{-1}\tfrac{1}{9}-\tan^{-1}\tfrac{1}{16}\ldots \text{ ad inf.},$$

which is the required result.

(II.) $$\sum_1^\infty\tan^{-1}\frac{1}{n^2}=\tan^{-1}\frac{S_1-S_3+S_5\ldots}{1-S_2+S_4\ldots}$$

where S_m = sum of products of such quantities as $1/n^2$ taken m together.

Now $$\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\ldots\equiv\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots$$
$$\equiv\theta\left[1-\frac{\theta^2}{\pi^2}S_1+\ldots+(-1)^n\frac{\theta^{2n}}{\pi^{2n}}S_n\ldots\right].$$

Hence, equating coefficients of θ^{2n+1}, $S_n/\pi^{2n}=1/(2n+1)!$; therefore

$$\sum_1^\infty\tan^{-1}\frac{1}{n^2}=\tan^{-1}\frac{\pi^2/3!-\pi^6/7!+\pi^{10}/11!\ldots}{1-\pi^4/5!+\pi^8/9!\ldots}=\tan^{-1}\frac{N}{D},\text{ say.}$$

Then $D+\iota N=1+x^2/3!+x^4/5!+x^6/7!\ldots$ $\quad[x=\pi(1+\iota)/\sqrt{2},$
$=\sinh x/x$ $\quad x^4=-\pi^4]$
$$=\frac{1-\iota}{\pi\sqrt{2}}(\sinh\pi/\sqrt{2}\cos\pi/\sqrt{2}+\iota\sin\pi/\sqrt{2}\cosh\pi/\sqrt{2}.)$$
$$N/D=(1-\tanh\pi/\sqrt{2}\cot\pi/\sqrt{2})/(1+\tanh\pi/\sqrt{2}\cot\pi/\sqrt{2}).$$

Now given series $=\tan^{-1}N/D-\frac{1}{4}\pi=\tan^{-1}(-\tanh\pi/\sqrt{2}\cot\pi/\sqrt{2})$.

15348. (Rev. T. WIGGINS, B.A.)—ABC is a triangle, and P is the middle point of BC. Draw AD at right angles to BC and from P draw PQ parallel to DA, so that Q and A are on the same side of BC. Let $2PQ=AB+AC$. Draw QR to meet DA in R, so that the angles PQR and QBP are equal. Show that RD is equal to one of the sides AB, AC.

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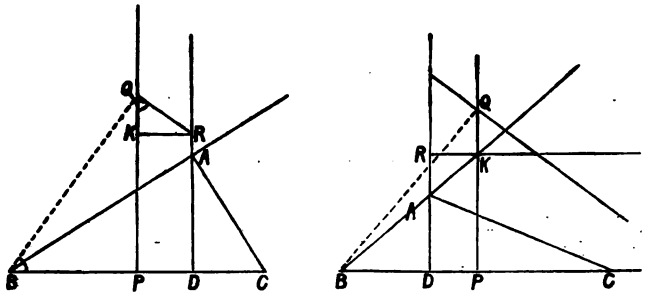
Solutions (I.) *by* R. Tucker, M.A.; (II.) *by* A. M. Nesbitt, M.A., W. H. L., *and* B. C. Wallis, F.C.P.

(I.) Through R draw RK parallel to BC; then the triangles RQK,



QBP are similar; therefore $QK/KR = BP/QP = a/(b+c)$, and

$$KR = (c^2 \sim b^2)/2a;$$

therefore $QK = (c^2 \sim b^2)/2(b+c) = \frac{1}{2}(c \sim b)$; therefore &c.

(II.) Since

$$\angle PQR = \angle QBP = \angle QCP,$$

QR is perpendicular to QC; therefore, if DV be parallel to QR, CP : PQ = VP : PD. Now

$$CD^2 - BD^2 = CA^2 - BA^2 = 2DP.BC.$$

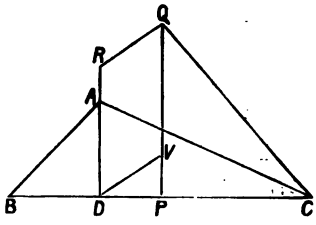
But $CA + AB = 2QP$,

$$CA - AB = 2(QP - AB);$$

whence

$$QP^2 - QP.AB = DP.PC = QP.PV;$$

therefore $QP - AB = PV$; therefore $QV = RD = AB$.



8712. (Professor Schoute.)—A surface S of the third order passing through two non-coplanar lines a and b is given, and a line l intersects a in A, b in B, and S for the third time in C, under the condition AC = CB. Find (1) the locus of the line l, and (2) its intersection with the given surface S.

Note by A. M. Nesbitt, M.A.

The plane which bisects at right angles the line which is perpendicular to both the given lines a, b will also bisect every line l which cuts both; hence the cubic surface S must break up into this plane and a quadric, *i.e.*, a hyperboloid of one sheet of which a and b are non-intersecting generating lines. It appears clear that a straight line which joins two random points on a couple of random generating lines (belonging, however, to the same system) of any given hyperboloid will be too unrestricted in its movements to trace out any locus at all; and, further, that its

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intersections with the given surface S will comprise every point in the two lines a, b and such points as are collinear with them in the plane referred to. If we take $y = mx$, $z = c$ and $y = -mx$, $z = -c$ for the lines a, b respectively, the "third intersections" of l and S will all lie in the plane of xy; and, moreover, if α, β be the x co-ordinates of A and B, the co-ordinates of C will be $\{\frac{1}{2}(\alpha+\beta), \frac{1}{2}[m(\alpha-\beta)], 0\}$. By properly choosing α and β (which are quite unrestricted) the point C may be made to take up any position at pleasure on the plane $z = 0$.

15379. (Rev. Prebendary WHITWORTH, M.A.)—In how many different orders can the eight digits 3, 3, 3, 3, 7, 7, 7, 7 be arranged so as to express a number divisible by 37?

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The following numbers evidently contain 37 as a factor—37, 333, 777, 703, 7733, 700003, 30000007, 30007.

Hence the following multiples of 37 can be formed:—(1) 37373737; (2) 77337733 (unique); 6 arrangements of 37, 333, 777; 3 arrangements of 37, 37, 7733; (1) 37773337; (2) 33737377 (unique)—*Total*, 13.

Note.—The Proposer remarks that the arrangement 37773337 is counted twice, for it is one of the arrangements of 37, 37, 7733, and is also mentioned by itself; he has also found six other arrangements which bring the total number up to 18, viz.: 33777337, 37337773, 37733377, 37737373, 73337737, 73737337.

15434. (A. ENSOR.)—Find the equation (E) whose roots are the ratios of the roots of $x^3 + px + q = 0$, and solve (E) when one root is given.

Solution by C. M. ROSS.

Put $f(x) \equiv x^3 + px + q = 0$. Let $y = \beta/\alpha$ = ratio of two roots; then, since $f(\beta) = 0$, $f(y\alpha) = 0$, also $f(\alpha) = 0$. Hence the required equation is obtained by eliminating α between $f(y\alpha) = 0$ and $f(\alpha) = 0$, *i.e.*, between $y^3\alpha^3 + py\alpha + q = 0$ and $\alpha^3 + p\alpha + q = 0$. By cross multiplication $\alpha^3/[pq(y-1)] = \alpha/[q(1-y^3)] = 1/[py(y^2-1)]$; therefore

$$q^2(y^2+y+1)^3 + p^3y^2(y+1)^2 = 0,$$

which is the required equation (E).

Again, on expanding (E), it becomes

$$q^2y^6 + 3q^2y^5 + (6q^2+p^3)y^4 + (7q^2+2p^3)y^3 + (6q^2+p^3)y^2 + 3q^2y + q^2 = 0,$$

i.e., $q^2(y^3 + 1/y^3) + 3q^2(y^2 + 1/y^2) + (6q^2+p^3)(y + 1/y) + 7q^2 + 2p^3 = 0$,

which is a reciprocal equation. Put $y + 1/y = z$, and equation reduces to

$$q^2z^3 + 3q^2z^2 + (3q^2+p^3)z + q^2 + 2p^3 = 0 \quad \text{.....(E')}.$$

Now, according to the Question, one root of (E) is known; but it is

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immaterial whether the root is considered of (E) or of (E′). Let it be of (E′); denote it by Q; then, if other roots are β and γ,

$$Q+\beta+\gamma=-3, \quad Q\beta+\beta\gamma+\gamma Q=(3q^2+p^3)/q^2 \ldots\ldots\ldots\text{(i., ii.)},$$
$$Q\beta\gamma=(q^2+2p^3)/q^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

From (i.) and (iii.), $\beta-\gamma=\{(3+Q)^2Qq^2+4(q^2+2p^3)\}^{\frac{1}{2}}/q\sqrt{Q}$; therefore

$$\beta=\{-(3+Q)q\sqrt{Q}+[(3+Q)^2Qq^2+4(q^2+2p^3)]^{\frac{1}{2}}\}/2q\sqrt{Q}$$

and $$\gamma=\{-(3+Q)q\sqrt{Q}-[(3+Q)^2Qq^2+4(q^2+2p^3)]^{\frac{1}{2}}\}/2q\sqrt{Q}.$$

Again, since $\beta+\gamma=\{(3q^2+p^3)Q+q^2+2p^3\}/Q^2q^2$ by (ii.)

$$=-(3+Q) \text{ by (i.)},$$

$$(3+Q)^2Qq^2+4(q^2+2p^3)=3q^2(Q^2+2Q+3)-Qp^3;$$

therefore the roots β and γ may be written

$$\{-(3+Q)q\sqrt{Q}\pm[3q^2(Q^2+2Q+3)-Qp^3]^{\frac{1}{2}}\}/2q\sqrt{Q}.$$

Again $x+1/x=Q$; therefore

$$x=\tfrac{1}{2}\{-Q\pm(Q^2-4)^{\frac{1}{2}}\} \quad \text{and} \quad x+1/x=\{-(3+Q)q\sqrt{Q}\pm m\}/2q\sqrt{Q};$$

$$\therefore\ x=\{-(3+Q)q\sqrt{Q}\pm m\pm[\{-(3+Q)q\sqrt{Q}\pm m\}^2-16q^2Q]^{\frac{1}{2}}\}/4q\sqrt{Q}.$$

Thus the roots of the equation (E) are $\frac{1}{2}[-Q\pm\sqrt{(Q^2-4)}]$

and $$1/4q\sqrt{Q}\{-(3+Q)q\sqrt{Q}\pm m\pm[\{-(3+Q)q\sqrt{Q}\pm m\}^2-16q^2Q]^{\frac{1}{2}}\},$$

where $$m^2\equiv 3q^2(Q^2+2Q+3)-Qp^3.$$

15125. (G. H. Hardy, M.A.)—Prove directly that

$$\lim_{k=1}\int_1^{1/k}\frac{du}{\sqrt{[(u^2-1)(1-k^2u^2)]}}=\tfrac{1}{2}\pi.$$

Solution by the Proposer.

Put $u^2=\cos^2\phi+(1/k^2)\sin^2\phi$, $u\,du=(1/k^2-1)\cos\phi\sin\phi\,d\phi$, when the integral beomes $\dfrac{1}{k}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{[\cos^2\phi+(1/k^2)\sin^2\phi]}}$, the limit of which for $k=1$ is $\frac{1}{2}\pi$. More generally, if $f(u)$ is continuous,

$$\lim_{k=1}\int_1^{1/k}\frac{f(u)\,du}{\sqrt{\{(u^2-1)(1-k^2u^2)\}}}=\tfrac{1}{2}\pi f(1).$$

[Indirectly the result is obvious, for the integral is known to be equal to K′.]

15437. (C. E. Youngman, M.A.)—Two circles (radii R, r and distance d from centre to centre) are so placed that $d^2=R^2+14Rr+r^2$ (r negative when $d<R$). Show that (1) any number of triads of circles mutually tangent can be drawn touching both these circles, the first at A, B, C, the second at A′, B′, C′; (2) $\Sigma\cot A(=k)$ and $\Sigma\cot A'(=k')$ are constants, connected by the relation $4(kk'\pm 3)=7(k\pm k')$. [When the two circles are equal $k=k'=2$ or $\sqrt{3}$; and when one of them opens into a straight line $k=\infty$ and $k'=1\frac{3}{4}$.]

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Solution by R. F. Davis, M.A.

If three circles of radius a touch each other two and two, their centres form an equilateral triangle of side $2a$. From the centroid of this triangle as centre can be described two circles of radii $a(2/\sqrt{3} \pm 1)$ each of which touches the three original circles, the points of contact for any one circle forming a similarly situated equilateral triangle. Hence it is always possible to describe an infinite number of triads of equal circles touching each other two and two, and also touching two given circles provided the latter are concentric and have their radii in the ratio

$$2+\sqrt{3} : 2-\sqrt{3}.$$

Invert from any point and there follows the porismatic relation as stated. The triangle ABC inverts into an equilateral triangle from either of the "isodynamic" points whose distance from the circumcentre $= R[\cot\omega \mp \sqrt{3})/(\cot\omega \pm \sqrt{3})]^{\frac{1}{2}}$ (Neuberg). Since this distance is constant and R is constant, so also is ω. The relation between the Brocard angles when $d > R$ is given by

$$[(k+\sqrt{3})(k'+\sqrt{3})]/[(k-\sqrt{3})(k'-\sqrt{3})] = [(2+\sqrt{3})/(2-\sqrt{3})]^2,$$

which reduces to $4(kk'+3) = 7(k+k')$. When $d < R$ (and R encloses r)

$$4(kk'-3) = 7(k-k').$$

Note on the Graphic Solution of Simultaneous Equations in three Unknowns.

By B. C. Wallis, F.C.P.

Take the x, y, and z axes at right angles and imagine the y axis swung downwards through a right angle so that YO and OZ are one straight line. Then an equation of the form $ax+by+cz=d$ is represented by a plane which cuts the plane of OX and OZ at the line $y=0$, $ax+cz=d$; the plane of OX and OY at the line $z=0$, $ax+by=d$; and the plane of OY and OZ at the line $x=0$, $by+cz=d$.

An equation of the form $x^2+y^2+z^2=a$ is represented by the surface of the sphere which has its centre at the origin and radius $\sqrt{a}$; and an equation $x^2-2ax+y^2-2by+z-2cz=d-a^2-b^2-c^2$ by the surface of a sphere centre (a, b, c) and radius $\sqrt{d}$.

An equation of the form $x^2+y^2=a$ is represented by the surface of a cylinder of which OZ is the axis and $\sqrt{a}$ the sectional radius; while $y^2+z^2=b$ has OX for axis and $\sqrt{b}$ as sectional radius.

Similarly an equation $x^2-2ax+z^2-2cz=d-a^2-c^2$ is represented by the surface of a cylinder having an axis parallel to OY through (a, o, c) and a sectional radius $\sqrt{a}$.

Using these principles, problems of forms of which the following three are examples may be worked graphically:—

$$2x+y+z=16, \quad x+2y+z=9, \quad x+y+2z=3 \quad \ldots\ldots\ldots \text{(A)};$$

$$x^2+y^2+z^2=25, \quad x+y+z=6, \quad 2x+y+z=8 \quad \ldots\ldots\ldots \text{(B)};$$

$$x^2+y^2=25, \quad x+y+z=8, \quad x+2y+3z=13 \quad \ldots\ldots \text{(C)}.$$

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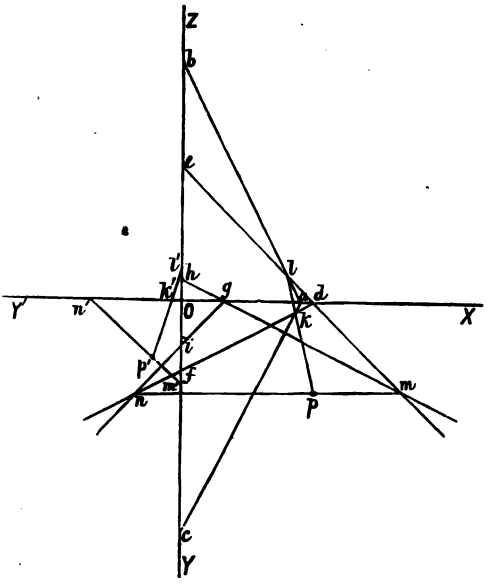

FIG. A.

Problem A.—$2x + y + z = 16$ is represented by *abc*, $x + 2y + z = 9$ by *def*, $x + y + 2z = 3$ by *ghi*. The intersection of *abc* and *def* is *lk*, of *ghi* and *def* is *mn*, of *lk* and *mn* is *p*. *p* gives $x = 9$. Imagine OY swung back and the whole swung through a right angle round OZ; then *l'k'* is the new position of *lk*, *m'n'* of *mn*, and *p'* of *p*. *p'* gives $y = 2$ and $z = -4$.

Solution—$x = 9$, $y = 2$, $z = -4$.

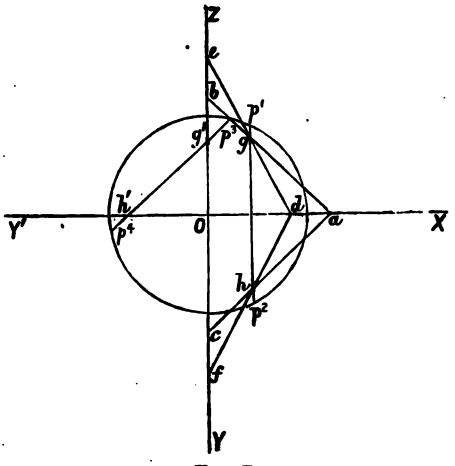

FIG. B.

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Problem B.—$x^2+y^2+z^2=25$ is represented by the circle, $x+y+z=6$ by *abe*, $2x+y+z=8$ by *def*. The intersection of *abc* and *def* is represented by gh and $g'h'$; the intersections of gh and $g'h'$ with the circle are represented by p', p^2, p^3, p^4; p' gives $x=2$, $z=4{\cdot}55$; p^2 gives $x'=2$, $y'=4{\cdot}55$; p_3 gives $y=-{\cdot}55$; p^4 gives $z'=-{\cdot}55$.

Solution—$x=2$, $y=4{\cdot}55$ or $-{\cdot}55$, $z=-{\cdot}55$ or $4{\cdot}55$.

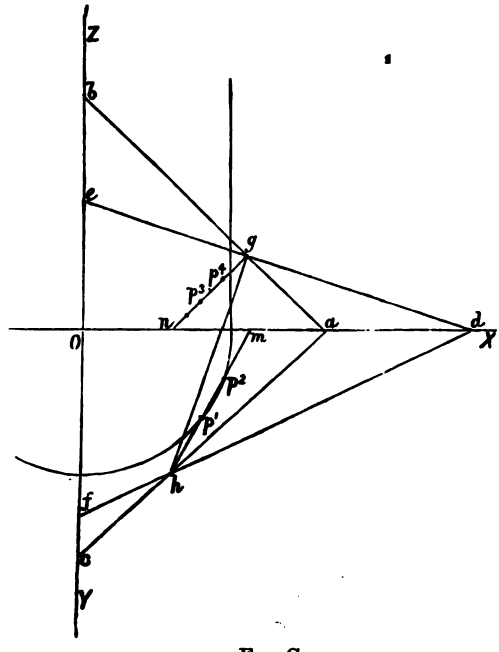

FIG. C.

Problem C.—$x^2+y^2=25$ is represented by the cylinder, $x+y+z=8$ by *abc*, $x+2y+3z=13$ by *def*. The intersection of *abc* and *def* is represented by gh, of which gn and hm are elevation and plan, respectively; gh cuts the cylinder in two points, of which p^3 and p^4 are the elevations and p' and p^2 the plans. p' gives $x=4$, $y=3$; p^2 gives $x'=4\frac{4}{5}$, $y'=1\frac{3}{5}$; p^3 gives $x=4$, $z=1$; p^4 gives $x'=4\frac{4}{5}$, $z'=1\frac{4}{5}$.

Solution—$x=4$ or $4\frac{4}{5}$, $y=3$ or $1\frac{3}{5}$, $z=1$ or $1\frac{4}{5}$.

15411. (Rev. T. ROACH, M.A.)—Prove that the series

$$\frac{1}{1.2.4.5}+\frac{1}{3.4.6.7}+\frac{1}{5.6.8.9}+\ldots=\frac{1}{36}.$$

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Solutions (I.) *by* LIONEL E. REAY, B.A., G. A. CARLE, M.A., B.Sc. (Edin.), *and others;* (II.) *by* H. J. WOODALL, A.R.C.S.

(I.) n-th term $= \dfrac{1}{(2n-1)\,2n\,(2n+2)\,(2n+3)}$. This can be written

$$\frac{1}{12}\left(\frac{1}{2n-1}-\frac{1}{2n+3}-\frac{1}{n}+\frac{1}{n+1}\right);$$

$\therefore\quad 12S = (\frac{1}{1}-\frac{1}{3}-\frac{1}{1}+\frac{1}{3})+(\frac{1}{3}-\frac{1}{7}-\frac{1}{3}+\frac{1}{5})+(\frac{1}{5}-\frac{1}{9}-\frac{1}{5}+\frac{1}{7})+\ldots = \frac{1}{3}$;

$\therefore\quad S = \frac{1}{36}$.

(II.) Series $= \dfrac{1}{3}\left(\dfrac{1}{1.5}-\dfrac{1}{2.4}+\dfrac{1}{3.7}-\dfrac{1}{4.6}+\dfrac{1}{5.9}-\dfrac{1}{6.8}+\ldots\right)$

$$= \frac{1}{3}\left[\frac{1}{4}\left(1-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}+\frac{1}{5}-\frac{1}{9}+\frac{1}{7}-\frac{1}{11}+\frac{1}{9}-\frac{1}{13}+\ldots\right)\right.$$
$$\left.-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{4}-\frac{1}{6}+\frac{1}{6}-\frac{1}{8}+\ldots\right)\right]$$
$$= \frac{1}{3}\left[\frac{1}{4}\left(1+\frac{1}{3}\right)-\frac{1}{2}\cdot\frac{1}{2}\right] = \frac{1}{36}.$$

15432. (ROBT. W. D. CHRISTIE.)—It is known that

$$p_n q_{n-1} - p_{n-1} q_n = \pm 1$$

where p_n, q_n are convergents of the equation $P^2 - NQ^2 = \pm 1$. Prove the same property for the remainders $r_n q_{n-1} - r_{n-1} q_n = \pm 1$. *E.g.*, $N = 73$.—$p_n = 8, 9, 17, 94, 487, 581, 1068$; $q_n = 1, 1, 2, 11, 57, 68, 125$; $r_n = 0, 0, 1, 6, 31, 37, 68$; then $68^2 - 37 \cdot 125 = -1$, $57 \cdot 37 - 68 \cdot 31 = +1$, ...

Solution by the PROPOSER.

We have $p_n q_{n+1} - q_n p_{n+1} = \pm 1$ (*v.* Todhunter's *Algebra*). Let

$$p_n = q_n t_n + r_n \quad \text{and} \quad p_{n+1} = q_{n+1} t_{n+1} + r_{n+1}.$$

Then $q_n q_{n+1}(t_n - t_{n+1}) + r_n q_{n+1} - q_n r_{n+1} = \pm 1$; but $t_n \equiv t_{n+1} \equiv a$ = integer of $\sqrt{p}$; therefore $r_n q_{n+1} - q_n r_{n+1} = \pm 1$, which proves the theorem.

15417. (R. TUCKER, M.A.)—PSp, QSq are two orthogonal focal vectors of a parabola. Show that the "S"-points of the four triangles PSQ, QSp, pSq, qSP lie on the circle $y^2 + (x-2a)^2 = 2a^2$.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by the* PROPOSER; (III.) *by* Professor SANJÁNA, M.A.

(I.) It may easily be shown by Adams' theorem, or by reciprocation, that PQ, Qp, pq, qP all touch an ellipse having the same focus and directrix, and whose excentricity $= 1/\sqrt{2}$. The feet of the perpendiculars from S upon these four tangents all lie on the auxiliary circle whose centre is

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at a distance 2AS from S and whose radius = 2AS$\sqrt{2}$. The "S"-points are the middle points of these perpendiculars; and therefore lie on a circle whose centre is at a distance AS from S and whose radius = AS$\sqrt{2}$.

(II.) The equation to PQ is $(m+m')y-2x=2amm'$, and the equation to the parallel through the "S"-point of PSQ is

$$(m+m')y-2x=a(mm'-1) \quad \text{..... (i.).}$$

The equation to the perpendicular to this is

$$2y+(m+m')x=a(m+m') \quad \text{..... (ii.).}$$

Now the condition for perpendicularity gives $2m.2m'=-(m^2-1)(m'^2-1)$; therefore $m^2m'^2+2mm'+1=m^2-2mm'+m'^2$; hence

$$mm'+1=\pm(m-m') \quad \text{..... (iii.).}$$

Now

$$m(y\mp a)+m'(y\pm a) = 2(x-a)$$

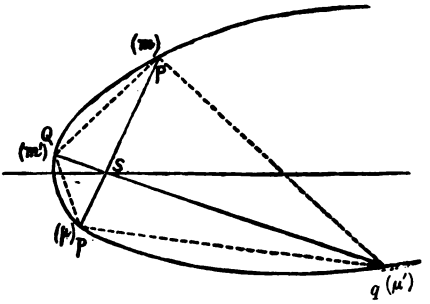
and

$$m(a-x)+m'(a-x)=2y;$$

whence

$$m\mu = am(a-x) = (x-a)^2+y^2+ay = \lambda+ay,$$

$$m'\mu = am'(a-x) = -[(x-a)^2+y^2-ay] = -(\lambda-ay).$$



This, with (iii.), gives

$$\mu^2-(\lambda^2-a^2y^2)=\mu(\mu n-\mu m')=2a\lambda\mu,$$

i.e., $y^2+(x-a)^2-a^2+2(a-x)a=0$ or $y^2+(x-2a)^2=2a^2$.

Similar results are obtained in the other cases.

(III.) The triangle SPQ being right-angled, its symmedian point Σ is the midpoint of the perpendicular SM from S on PQ. Let

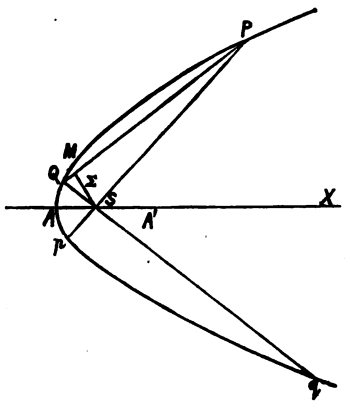
$$\angle PSX=\theta, \quad AS=a.$$

Then $SP=2a/(1-\cos\theta)$,

$SQ=2a/(1+\sin\theta)$;

and

$$S\Sigma=\tfrac{1}{2}SM=\tfrac{1}{2}\frac{SP.SQ}{\sqrt{(SP^2+SQ^2)}} = \frac{a}{\sqrt{(3-2\cos\theta+2\sin\theta)}}.$$



The "m" of

$$PQ=\frac{SP\sin\theta-SQ\cos\theta}{SP\cos\theta+SQ\sin\theta} = \frac{1-\cos\theta+\sin\theta}{\cos\theta+\sin\theta};$$

hence the "m" of

$$SM=-\frac{\cos\theta+\sin\theta}{1-\cos\theta+\sin\theta};$$

so that, if $S\Sigma$ is inclined at ϕ to the axis,

$$\sin\phi=\frac{\cos\theta+\sin\theta}{\sqrt{(3-2\cos\theta+2\sin\theta)}}, \quad \cos\phi=-\frac{1-\cos\theta+\sin\theta}{\sqrt{(3-2\cos\theta+2\sin\theta)}}.$$

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Thus for Σ the co-ordinates are

$$x_1 = S\Sigma\cos\phi = \frac{-a(1-\cos\theta+\sin\theta)}{3-2\cos\theta+2\sin\theta},$$

$$y_1 = S\Sigma\sin\phi = \frac{a(\cos\theta+\sin\theta)}{3-2\cos\theta+2\sin\theta},$$

the focus being the origin. Hence

$$y_1^2+(x_1-a)^2$$
$$= a^2[(\cos\theta+\sin\theta)^2+(4-3\cos\theta+3\sin\theta)^2]+(3-2\cos\theta+2\sin\theta)^2$$
$$= \frac{a^2(1+2\sin\theta\cos\theta+16-24\cos\theta+24\sin\theta+9-18\sin\theta\cos\theta)}{(3-2\cos\theta+2\sin\theta)^2}$$
$$= 2a^2;$$

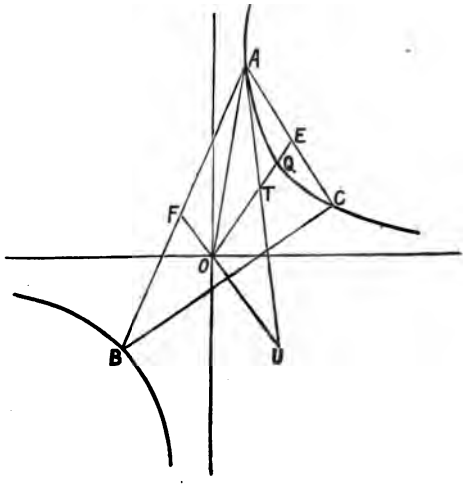
so that Σ lies on a circle whose centre is A′ where SA′ = SA. Changing θ to $\theta+\frac{1}{2}\pi$, $\theta+\pi$, $\theta+\frac{3}{2}\pi$, the co-ordinates of the other symmedian points are obtained, and they are seen to lie on the same circle

$$y^2+(x-2a)^2 = 2a^2.$$

15420. (Professor Neuberg.)—Construire une hyperbole équilatère, connaissant trois points A, B, C de la courbe et la tangente menée en A.

Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) Let AT be the given tangent, which for the sake of simplicity may be supposed to fall within the angle BAC; E, F the middle points of AC, AB. Upon EA, FA describe internally segments of circles containing angles respectively equal to EAT, FAT. Their intersection O will determine the centre of the curve lying on the nine-points circle of ABC



(for AOE + AOF = EAT + FAT = A). For (referring to figure)

$$EA^2 = OE^2-OQ^2 = OE^2-OT\,.\,OE = OE\,.\,TE,$$

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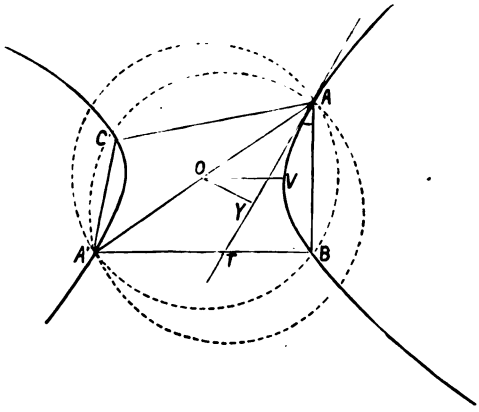
and therefore EA touches at A the circum-circle of AOT, and EAT = AOE. Similarly, FA touches at A the circum-circle of AOU, and FAT = AOF.

(II.) Let AT be the tangent at A. On AB describe a segment of a circle containing an angle = BAT; on AC a segment containing an angle = CAT; then A′ is the extremity of the diameter through A. (See Besant, chap. v.)



Bisect AA′ in O, the centre; draw OY perpendicular to AT; bisect the angle AOY by OV, and take OV the mean proportional between OA, OY: then V is the extremity of the transverse axis (Besant). The hyperbola can now be easily constructed.

15052. (R. Tucker, M.A.)—P, Q, L (m_1, m_2, m_3) are co-normal points on a parabola, as also are R, T, L′ (μ_1, μ_2, μ_3), for which O, O′ are the respective co-normal points. PQ, RT are also orthogonal focal chords. Prove that OO′ and LL′ pass through fixed points on the axis and intersect on a fixed perpendicular to the axis. The points O, O′ lie on a parabola.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by* A. W. Poole.

(I.) The normals at P, Q are orthogonal, and also those at R, T; hence O, O′ lie on the parabola $y^2 = a(x-3a)$.

Let $\angle PSX = \alpha$; then $SP = 2a/(1-\cos\alpha)$; also $SR = 2a/(1+\sin\alpha)$, $SQ = 2a/(1+\cos\alpha)$, $ST = 2a/(1-\sin\alpha)$. The ordinates of P, Q are

$$y_1 = 2a\sin\alpha/(1-\cos\alpha), \quad y_2 = -2a\sin\alpha/(1+\cos\alpha);$$

hence, if y_3 be the ordinate of L, $y_3 = -(y_1+y_2) = -4a\cot\alpha$. So also the ordinate of L′, $y'_3 = 4a\tan\alpha$. Hence the equation of LL′, $y(y_3+y'_3) = 4ax + y_3y'_3$, becomes $y(\tan\alpha - \cot\alpha) = x - 4a$, or

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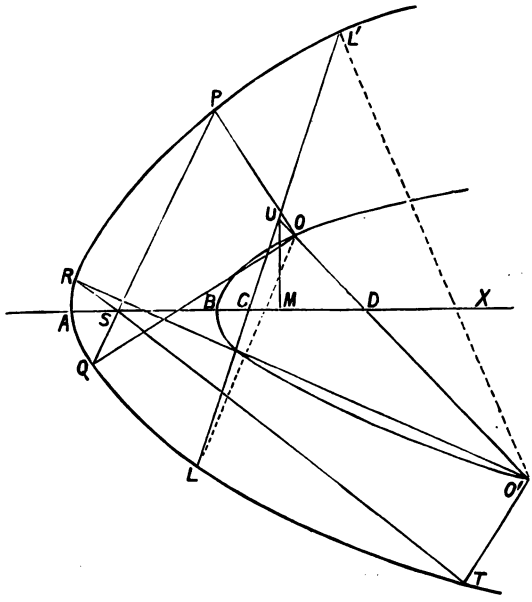
$y = -\frac{1}{2}\tan 2\alpha\,(x-4a)$, which shows that LL′ goes through the fixed point $(4a,\ 0)$.



The ordinate of O is evidently $\frac{1}{2}(y_1+y_2) = 2a\cot\alpha$; therefore its abscissa, $y^2/a+3a$, is $4a\cot^2\alpha+3a$. Similarly, the ordinate of O′ is $-2a\tan\alpha$, and its abscissa $4a\tan^2\alpha+3a$. Hence the equation of OO′ is

$$(y+2a\tan\alpha)\,4a\,(\cot^2\alpha-\tan^2\alpha) = 2a\,(\cot\alpha+\tan\alpha)\,(x-4a\tan^2\alpha-3a),$$

i.e., $2y\,(\cot\alpha-\tan\alpha) = x-7a$, or $y = \frac{1}{4}\tan 2\alpha\,(x-7a)$, which shows that OO′ goes through the fixed point $(7a,\ 0)$.

Eliminating α from the equations of LL′, OO′, we get $x = 5a$, the locus of their intersection U. In the figure $AB = 3a$, $AC = 4a$, $AM = 5a$, $AD = 7a$.

(II.) The normals at P, Q, L are concurrent; therefore

$$m_1+m_2+m_3 = 0.$$

The co-ordinates of O are found to be $x = 2a+a\,(m_1^2+m_1m_2+m_2^2)$, $y = am_1m_2m_3$. The equation of PQ is

$$\frac{(x-am_1^2)}{(m_2^2-m_1^2)} = \frac{(y-2am_1)}{(2m_2-2m_1)}$$

or $2\,(x-am_1^2) = (y-2am_1)(m_1+m_2)$ or $-(m_1+m_2)y+2x+2am_1m_2 = 0$ or $m_3y+2x+2am_1m_2 = 0$. But PQ is a focal chord; therefore $2a+2am_1m_2 = 0$; therefore $m_1m_2 = -1$, and also $\mu_1\mu_2 = -1$. But PQ is perpendicular to RT, *i.e.*, $m_3y+2x = 0$ is perpendicular to $\mu_3y+2x = 0$; therefore

$$m_3\mu_3+4 = 0;$$

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therefore the co-ordinates of O are now

$$x = a[2 + m_1^2 + m_1 m_2 + m_2^2]$$
$$= a[2 + m_1^2 - 1 + m_2^2] = a[1 + m_1^2 + m_2^2] = a[3 + m_3^2],$$
$$y = am_1 m_2 m_3 = -am_3.$$

Similarly, the co-ordinates of O′ are $x = a(\mu_3^2 + 3)$, $y = -a\mu_3$. So, by eliminating m_3 and μ_3 respectively, we get that O, O′ lie on the same parabola, which is $x = a(y^2/a^2 + 3)$ or $ax = y^2 + 3a^2$.

The equation of OO′ is

$$\frac{x - a(\mu_3^2 + 3)}{m_3^2 - \mu_3^2} = \frac{y + a\mu_3}{-m_3 + \mu_3} \quad \text{or} \quad \frac{x - a(\mu_3^2 + 3)}{m_3 + \mu_3} = \frac{y + a\mu_3}{-1};$$

therefore $-x + 3a = y(m_3 + \mu_3) + a\mu_3 m_3$; therefore this line meets $y = 0$ in the point $x = 3a - a\mu_3 m_3 = 3a + 4a = 7a$.

The equation of LL′ is

$$(x - am_3^2)/(\mu_3^2 - m_3^2) = (y - 2am_3)/(2\mu_3 - 2m_3)$$

or $2(x - am_3^2) = (\mu_3 + m_3)(y - 2am_3)$ or $2x = (\mu_3 + m_3)y - 2am_3\mu_3$; therefore this line meets $y = 0$ in the point $x = -am_3\mu_3 = 4a$. Also the lines OO′, LL′ intersect where $x = 5a$ (by eliminating y).

15413. (Professor Sanjána, M.A.)—Prove the following three expressions equal, r being any positive integer,

$$\frac{(r-1)!}{(1+x)^r}\left\{x^{r-1} - rx^{r-2} + \frac{r(r-1)}{1.2}x^{r-3} + \ldots + r\right\},$$

$$(-1)^{r-1}\frac{(r-1)!}{(1+x)^r}\left\{x^{r-1} - rx^{r-2}(1+x) + \frac{r(r-1)}{1.2}x^{r-3}(1+x)^2 - \ldots + (-1)^{r-1}r(1+x)^{r-1}\right\},$$

$$\frac{(r-1)!}{(1+x)^r}\left\{1 + (1+x) + (1+x)^2 + \ldots + (1+x)^{r-1}\right\}.$$

Solution by L. E. Reay, B.A., J. Blaikie, M.A., C. M. Ross, *and others.*

$\left\{x^{r-1} + rx^{r-2} + \frac{r(r-1)}{1.2}x^{r-3} + \ldots + r\right\}$ may be written $\frac{(1+x)^r - 1}{x}$.

$$(-1)^{r-1}[x^{r-1} - rx^{r-2}(1+x) + \ldots]$$
$$= (-1)^{r-1}\left\{\frac{[x - (1+x)]^r + (-1)^{r-1}(1+x)^r}{x}\right\} = \frac{(1+x)^r - 1}{x}$$

Now $\frac{(1+x)^r - 1}{x} = \frac{1 - (1+x)^r}{1 - (1+x)} = 1 + (1+x) + (1+x)^2 + \ldots (1+x)^{r-1}$.

Hence all three expressions may be reduced to the same form.

N.B.—Mr. Blaikie suggests that the common factor $(r-1)!/(1+x)^r$ might well be dispensed with.

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15414. (Robt. W. D. Christie.)—In any right-angled triangle where the legs a, b differ by unity, and c, the hypotenuse, is an integer, prove $2+3c+4a = x^2+y^2$, and find an infinite number of integral values of a, b, c, x, and y.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E., *and* Professor Sanjána, M.A.; (II.) *by* J. Blaikie, M.A., *and* R. F. Davis, M.A.; (III.) *by the* Proposer.

(I.) Let $a < b$; then $b = a+1$, $a+b = 2a+1 = d$ (suppose),

$$c^2 = a^2+b^2 = 2a^2+2a+1;$$

therefore $\quad 2c^2 = (2a+1)^2+1 = d^2+1, \quad$ or $\quad d^2-2c^2 = -1.$

The solutions (c, d) of this Diophantine are known to be as below, whence follow a, b:

$c =$	1	5	29	169	985	...
$d =$	1	7	41	239	1393	...
$a =$	0	3	20	119	696	...
$b =$	0	4	21	120	697	...

where each solution (c_r, d_r) may be derived from the two preceding in succession ad inf. by the relation $c_r = 6c_{r-1}-c_{r-2}$, $d_r = 6d_{r-1}-d_{r-2}$.

Another relation is $c_r = 3c_{r-1}+2d_{r-1}$, whence $3c_{r-1}+4a_{r-1}+2 = c_r$. Now c_r is known to be of form $(x_r^2+y_r^2)$ always; therefore

$$3c+4a+2 = x^2+y^2 \text{ always.}$$

By resolving c_r, the successive x_r, y are found:

$c =$	1	5	29	169	985	...
$x =$	1	2	5	12	29	...
$y =$	0	1	2	5	12	...

and each x_r, y_r may be derived from the two preceding in succession ad inf. by the relation $x_r = 2x_{r-1}+x_{r-2}$, $y_r = 2y_{r-1}+y_{r-2}$.

(II.) Let y, z be two integers which satisfy the relation

$$2yz-1 = y^2-z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then y', z' will also satisfy the equation, if $y' = 5y+2z$, $z' = 2y+z$. For, substituting, we obtain $2(5y+2z)(2y+z)-1 = (5y+2z)^2-(2y+z)^2$, or

$$2(10y^2+9yz+2z^2)-1 = (7y+3z)(3y+z) = 21y^2+16yz+3z^2,$$

which reduces to (1).

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If we take $a = y^2 - z^2$, $b = 2yz$, $c = y^2 + z^2$, then $a^2 + b^2 = c^2$, and $a = b - 1$ by (1). Also

$$2 + 3c + 4a = 2(2yz - y^2 + z^2) + 3(y^2 + z^2) + 4(y^2 - z^2) = 5y^2 + 4yz + z^2 = x^2 + y^2,$$

when $x = 2y + z$. If we substitute y', z' for y, z, we find that

$$x' = 2y' + z' = 12y + 5z = 5x + 2y, \quad y' = 5y + 2z = 2x + y, \quad z' = x.$$

Equation (1) is satisfied by the values $y = 2$, $z = 1$, which give $x = 5$, $a = 3$, $b = 4$, $c = 5$, $2 + 3c + 4a = 29 = 5^2 + 2^2$. The following table gives the first three cases, and an infinite number may be derived by means of the relations established above :—

x	y	z	a	b	c	$2 + 3c + 4a$
5	2	1	3	4	5	29
29	12	5	119	120	169	985
169	70	29	4059	4060	5741	33461

(III.) Let q_n be the n-th convergent of Y in the equation

$$X^2 - 2Y^2 = \pm 1;$$

then $$4(q_n^2 - q_{n-1}^2) + 3(q_n^2 + q_{n-1}^2) + 2 = (2q_n + q_{n-1})^2 + q_n^2 \quad \ldots\ldots\ldots\ldots (1)$$

or $$4(2q_n q_{n-1}) + 3(q_n^2 + q_{n-1}^2) + 2 = (2q_n + q_{n-1})^2 + q_n^2 \quad \ldots\ldots\ldots\ldots (2).$$

In the theorem let a or $b = q_n^2 - q_{n-1}^2$ or $2q_n q_{n-1}$,

$$c = q_n^2 + q_{n-1}^2, \quad x = 2q_n + q_{n-1}, \quad y = q_n.$$

For the connexion *v.* Dr. Muir, *Proc. Roy. Soc. Edin.*, p. 264.

10113. (Professor Gob).—Lorsque la base BC d'un triangle ABC est fixe et que l'angle de Brocard est constant, les centres des cercles de Neuberg (N_b), (N_c) décrivent une circonférence ; le cercle (N_b) a pour enveloppe un limaçon de Pascal ; la droite de Lemoine de ABC enveloppe une conique ayant pour foyers les points B, C.

Solutions (I.) *by* C. E. Youngman, M.A. ; (II.) *by* R. Tucker, M.A. ; (III.) *by* Professor Sanjána, M.A.

(I.) Draw AD, CF touching the circle ABC, and C*f*, A*d* parallel to them, the points D, *d* being on BC and F, *f* on AB. Then FD is the Lemoine line, and *fd* is parallel to it ; also the triangles AC*f*, *d*AC have angles A, B, C, and therefore the circle B*fd* is (N_b). Let B′ be the point opposite to B on this circle.

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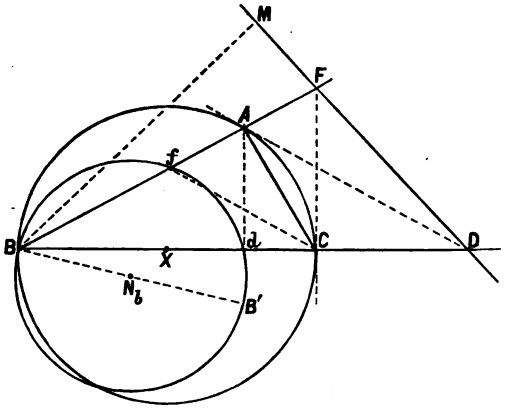
The triangle N_bCA is isosceles, with vertical angle 2ω; therefore $CN_b = \frac{1}{2}b \operatorname{cosec}\omega$. And the radius of the circle N_b (which is orthogonal to the circle with centre C and radius b) is $\frac{1}{2}b \operatorname{cosec}\omega \sqrt{(1-4\sin^2\omega)}$, or $e\,CN_b$, if we write e for $\sqrt{(1-4\sin^2\omega)}$. Therefore $BN_b : CN_b = \text{const.}$, and N_b describes a circle whose centre divides BC externally in the ratio $e^2 : 1$. Its radius is $\frac{1}{2}ae\operatorname{cosec}^2\omega$. And naturally N_c describes an equal circle, symmetrical on BC to the other. The locus of B′ is also a circle.



Invert the figure from B; then (N_b) becomes a straight line n_b, and B′ becomes b', the projection of B on n_b; therefore the locus of b' is a circle. Hence the envelope of n_b is a conic having B for focus, and the envelope of (N_b) is the focal inverse of this—a limaçon.

Draw BM perpendicular to FD, and bisect BC at X. Then BfB′, BMD are similar triangles, and $BM \cdot BB' = Bf \cdot BD = BC \cdot BA$; therefore $BM \cdot BN_b = BX \cdot BA$, and the triangles BMX, BAN_b are similar; therefore $XM : XB = AN_b : BN_b = 1 : e$ and M moves in a circle about X. Hence FD touches an ellipse with focus B, centre X, and eccentricity BX : XM or e; which is also the eccentricity of the Brocard ellipse.

(II.) Reference is made to Casey's *Conics* (1893, p. 424), $C_1N_c = \frac{1}{2}c\cot\omega$, $(\text{radius})^2 = \frac{1}{4}[c^2(\cot^2\omega - 3)]$.

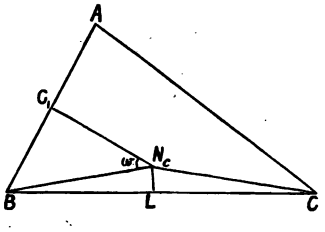
Taking BC and the perpendicular through B to BC as axes, we get

$X = BL = c\sin(B+\omega)/2\sin\omega$,

$Y = N_cL = -c\cos(B+\omega)/2\sin\omega$,

whence

$X^2 + Y^2 = BN_c^2 = \frac{1}{4}c^2\operatorname{cosec}^2\omega$ ... (i.).



Now

$$2aX = ac\sin(B+\omega)/\sin\omega$$
$$= ac(c^2+a^2)\sin B/\lambda\sin\omega$$
$$= c^2 + a^2;$$

therefore (i.) becomes $X^2 + Y^2 = \frac{1}{4}\operatorname{cosec}^2\omega\,(2aX - a^2)$. Again, referring the Neuberg circle (N_c) to the same axes, we have

$$X^2 + Y^2 = [c\sin(B+\omega)\,X]/\sin\omega + [c\cos(B+\omega)\,Y]/\sin\omega + c^2 = 0.$$

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This shows, as in Casey, that the circle passes through C and also cuts BC in L′, so that $BL' = c^2/a$. Refer now the circle to parallel axes through C and our equation becomes

$$X^2 + Y^2 = [(a^2-c^2)a]X = 2Y\sqrt{\{\tfrac{1}{4}c^2(\cot^2\omega - 3) - [(a^2-c^2)/2a]\}};$$

therefore $CL' = (a^2-c^2)/a$. Arranging the equations by powers of c we have

$$Ac^4 + Bc^2 + D = 0,$$

where $A = (X^2+Y^2)/a^2$, $B = 2(X^2+Y^2)(a-2X) - (\cot^2\omega - 3)Y^2$,

and $D = (X^2+Y^2)[(X-a)^2+Y^2]$. For the envelope we have, putting $\cot^2\omega - 3 = \lambda$,

$$[2(X^2+Y^2)(a-X) + a\lambda Y^2]^2 = 4(X^2+Y^2)[(X-a)^2+Y^2]$$

i.e.,
$$4(X^2+Y^2)^2 - 4a^2\lambda(X^2+Y^2) + 4a\lambda X(X^2+Y^2) = a^2\lambda Y^2.$$

The Lemoine line (see Simmons, p. 133) is the polar of the Lemoine point with respect to the circum-circle, *i.e.*, its equation is $bc\alpha + ca\beta + ab\gamma = 0$. The perpendiculars from B and C are (Whitworth, p. 48)

$$[(ca.2\Delta)/b]/(\Sigma a^2b^2 - 2abc\,\Sigma a\cos A + \Sigma a^4)^{\frac{1}{2}}$$

and
$$[(ab.2\Delta)/c]/(\Sigma a^2b^2 - 2abc\,\Sigma a\cos A + \Sigma a^4)^{\frac{1}{2}};$$

therefore the product

$$= 4a^2\Delta^2/\Sigma(a^4 - b^2c^2) = 4a^2\Delta^2/\Sigma b^2c^2(2\cos 2\omega - 1) = a^2\sin^2\omega/(2\cos 2\omega - 1).$$

Hence the envelope is a central conic.

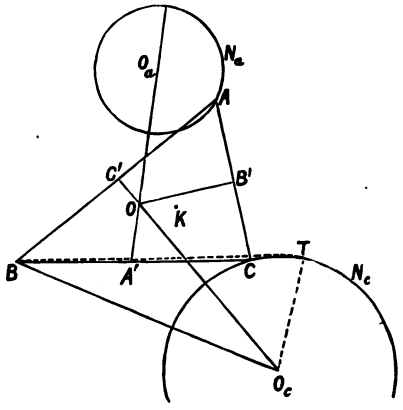
(III.) Refer to BC and its perpendicular bisector A′O as axes of x and y. Then N_a is

$$x^2 + y^2 - ay\cot\omega + \tfrac{1}{4}a^2 = 0.$$

So N_c referred to AB and C′O is

$$x^2 + y^2 - cy\cot\omega + \tfrac{1}{4}c^2 = 0;$$

hence $C'O_c = \frac{1}{2}c\cot\omega$, and $BC' = \frac{1}{2}c$; therefore the triangle $BC'O_c$ is fixed in species. The locus of A being a circle, and B being a fixed point, it is evident that the locus of C′ is also a fixed circle. Thus of the triangle $BC'O_c$ one vertex is fixed and a second moves on a circle; therefore the third O_c also moves on a fixed circle. So also for O_b.



The tangent from B to N_c equals BA or c; and the radius of N_c is $\frac{1}{2}c\sqrt{(\cot^2\omega - 3)}$; so that TO_c : BT is a fixed ratio, and therefore TO_c : BO_c is also fixed. Thus the circle N_c is described by taking points on the fixed locus of O_c as centres, and by lengths proportional to the distances of these points from the fixed point B as radii. It hence follows that the envelope of N_c is a limaçon. So also for N_b.

The co-ordinates of K, the symmedian point, are $\frac{1}{2}a\tan\omega$ (cot B − cot C),

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$\frac{1}{2}a\tan\omega$; the Lemoine line is the polar of this for the circum-circle, *i.e.*, for $x^2+y^2-ay\cot A-\frac{1}{2}a^2=0$. Thus the equation of the line is

$$x(\cot B-\cot C)+y(1-\cot A\cot\omega)-\tfrac{1}{2}a(\cot A+\cot\omega)=0.$$

The product of the perpendiculars from B and C on this is found to be

$$\frac{a^2(\cot A+\cot B)(\cot A+\cot C)}{(\cot B-\cot C)^2+(1-\cot A\cot\omega)^2}=\frac{a^2\operatorname{cosec}^2 A}{\cot^2 B+\cot^2 C-1+\cot^2 A(\Sigma\cot^2 A)}$$

$$=\frac{a^2\operatorname{cosec}^2 A}{(\cot^2 A+1)(\Sigma\cot^2 A-1)}=\frac{a^2}{\cot^2\omega-3}.$$

Thus the envelope of the Lemoine line is an ellipse having B and C for foci.

15383. (A. M. Nesbitt, M.A.)—If $\alpha+\beta+\gamma$ be a multiple of π, and if $x^r\sin r\alpha+y^r\sin r\beta+z^r\sin r\gamma$ vanishes when $r=1$ and $r=2$, it vanishes for any positive integral value of r.

Solution by the Proposer.

If we take $x\sin\alpha=q-r$, $y\sin\beta=r-p$, $z\sin\gamma=p-q$, we may put $x\cos\alpha=\lambda p+\mu$, $y\cos\beta=\lambda q+\mu$, $z\cos\gamma=\lambda r+\mu$, in which λ, μ may at present have any values. These yield

$$x(\cos\alpha+\lambda\sin\alpha)=\lambda(p+q-r)+\mu=\lambda q+\mu-\lambda y\sin\beta=y(\cos\beta-\lambda\sin\beta) \quad \ldots\ldots (1).$$

So also
$$y(\cos\beta+\lambda\sin\beta=z(\cos\gamma-\lambda\sin\gamma) \quad \ldots\ldots\ldots\ldots (2),$$

$$z(\cos\gamma+\lambda\sin\gamma)=x(\cos\alpha-\lambda\sin\alpha) \quad \ldots\ldots\ldots\ldots (3).$$

By multiplying (1), (2), and (3) we get $\lambda^2+\Sigma\cot\beta\cot\gamma=0$; therefore $\lambda^2=-1$. Hence $x(\cos\alpha+i\sin\alpha)=y(\cos\beta-i\sin\beta)$, ...; therefore $\Sigma x^n(\cos n\alpha+i\sin n\alpha)=\Sigma x^n(\cos n\alpha-i\sin n\alpha)$; therefore $\Sigma x^n\sin n\alpha=0$.

15357. (A. M. Nesbitt, M.A.) — Evaluate $\int_0^\infty\frac{x\,dx}{e^{cx}\cosh x}$, c being between $+1$ and -1.

Solution by G. H. Hardy, M.A.

Since $e^{-cx}\operatorname{sech}x=2\sum_0^\infty(-)^n e^{-(c+2n+1)x}$ and $\int_0^\infty e^{-px}x\,dx=\frac{1}{p^2}$ $(p>0)$,

$$\int_0^\infty\frac{x\,dx}{e^{cx}\cosh x}=2\sum_0^\infty\frac{(-)^n}{(c+2n+1)^2}=2\frac{d^2}{dc_2}\log[\Gamma\tfrac{1}{4}(c+1)/\Gamma\tfrac{1}{4}(c+3)].$$

By integration with respect to c we can deduce the values of

$$\int_0^\infty\frac{e^{-cx}}{\cosh x}dx,\quad\int_0^\infty\frac{e^{-ax}-e^{-bx}}{\cosh x}\frac{dx}{x}.$$

It is not necessary that $c<1$; the condition $c>-1$ is sufficient.

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5225. (The late T. COTTERILL, M.A.)—If the sides ab, $a'b'$ of two triangles abc, $a'b'c'$ meet in z and a transversal cut ca, cb, $c'a'$, $c'b'$ respectively in α, β, α', β', prove that the two conics $zab'a\beta'$, $za'b\alpha'\beta$ cut again in three points on the cubic locus of points at which the three pairs of points (aa'), (bb'), (cc') subtend angles in involution.

Solution by Professor NANSON.

The locus of the intersection of the two conics, for different positions of the transversal, is a cubic. For, if x, y, γ, x', y', γ' are the lines ca, $c'b'$, ab', $c'a'$, cb, $a'b$ and ω is the variable transversal, the two conics are $xy = k\gamma\omega$, $x'y' = k'\gamma'\omega$, where k, k' are constants determined by the fact that the conics pass through z. Eliminating ω, the locus of the remaining three intersections is the cubic $k'xy\gamma' = kx'y'\gamma$. This cubic passes through the nine intersections of x, y, γ' with x', y', γ. The involution cubic also obviously passes through these nine points. Now, if p be the intersection, distinct from c, c', z, of the conics corresponding to the position cc' of the transversal ω, we clearly have

$$p(ab'cc') = z(abcc) = z(a'bc'c) = p(a'bc'c)$$

and therefore p is also a point on the involution cubic. Hence the two cubics coincide.

15462. (R. CHARTRES.)—If $1/x = \cdot\dot{a}bod\ldots h\dot{k}$ in any scale, radix $= r$, then $p/x = \cdot\dot{k}abcd\ldots\dot{h}$. Find p, and show that there is an infinite number of scales where $1/x$ gives the same number of digits as above, and where p/x can be obtained by putting the last digit first. Give also the general value for the radix.

Solution by the PROPOSER.

$1/(mr-1)$ can be expressed as a recurring radix fraction by putting unity for the last digit and multiplying by m towards the left; therefore when $1/x = k/(mr-1)$ the fraction $= \cdot\dot{a}bc\ldots h\dot{k}$ and $m/x = \cdot\dot{k}abc\ldots\dot{h}$. Also, since $kx = mr-1$, therefore $1/x = (k+mn)/[m(r+nx)-1]$ will have in the scale, radix $(r+nx)$, the last digit $(k+mn)$ and m for multiplier, and m/x will be found by putting the last digit first, thus: $\frac{1}{7}$ with radix 3, 10, 17, $\ldots = \cdot\dot{0}10212$, $\cdot\dot{1}42857$, $\cdot\dot{2}74(14)9(12)$, $\ldots$, the multiplier in each case being 5, and $\frac{4}{7}$ will be found at once by putting the last digit first.

15469. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Find a series of pairs of right-angled triangles whose sides $(a, b, c; a', b', c)$ are integers, and with a common hypothenuse (c), and such that $c-b = b-b' = 1$.

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Solutions (I.) *by* Rev. T. Wiggins, B.A., *and* Professor Sanjána, M.A.; (II.) *by* D. Biddle; (III.) *by the* Proposer.

(I.) The sides are $[\sqrt{(2b+1)}, b, b+1]$, $[2\sqrt{b}, b-1, b+1]$; therefore $\sqrt{(2b+1)}$, $\sqrt{b}$ must be integral. Let $2b+1 = x^2$, $b = y^2$; therefore $x^2-2y^2 = 1$, which is satisfied by $x = 3$, $y = 2$. Hence

$$(x^2-2y^2) = (3^2-2.2^2)^n;$$

therefore $$(x+\sqrt{2}y)(x-\sqrt{2}y) = (3+2\sqrt{2})^n(3-2\sqrt{2})^n.$$

Now put $x+\sqrt{2}y = (3+2\sqrt{2})^n$ and $x-\sqrt{2}y = (3-2\sqrt{2})^n$; therefore $2x = (3+2\sqrt{2})^n+(3-2\sqrt{2})^n$, $2\sqrt{2}y = (3+2\sqrt{2})^n-(3-2\sqrt{2})^n$. As n may have any value, we find the series of values of x and y, whence we obtain b.

Cor.—$n = 1$, $y = 2$, $b = 4$; therefore triangles are (3,4,5), (4, 3, 5). $n = 2$, $y = 12$, $b = 144$; therefore triangles are (17, 144, 145), (24, 143, 145). $n = 3$, $y = 70$, $b = 4900$; therefore triangles are (99, 4900, 4901), (140, 4899, 4901).

(II.) Here $a^2+b^2 = (b+1)^2 = a'^2+(b-1)^2$, whence $a^2 = 2b+1$ and $a'^2 = 4b$. Therefore $b = \square$, always of form $4m$, for, if it were of form $4m+1$, a^2 would be of form $4m-1$, which is impossible. Let $b = 4x^2$; then $a' = 4x$ and $a^2 = 8x^2+1$. Consequently, if a, which is always odd, be represented by $2n+1$, we have $x^2 = \frac{1}{2}n(n+1)$, showing that it belongs to the third order of polygonal numbers, namely, 1, 3, 6, 10, 15, 21, 28, 36, ..., but only to such as are square. Now, n and $n+1$ cannot have any factor in common; yet $\frac{1}{2}n(n+1)$ is composed of factors that differ. Therefore $x^2 = \lambda^2\mu^2$, where $\lambda^2 = \frac{1}{2}n$ or $\frac{1}{2}(n+1)$, the respective values of μ^2 being $n+1$ and n. Thus $\mu^2 = 2\lambda^2\pm1$, according as n is even or odd. Consequently $x = \lambda(2\lambda^2\pm1)^{\frac{1}{2}}$ and $a^2 = 16\lambda^4\pm8\lambda^2+1$, whence $a = 4\lambda^2\pm1 = 2n+1$, giving $n = 2\lambda^2$ or $2\lambda^2-1$; $b = 4\lambda^2(2\lambda^2\pm1)$, $c = (2\lambda^2)^2+(2\lambda^2\pm1)^2$, whilst $a' = 4\lambda(2\lambda^2\pm1)^{\frac{1}{2}}$ and $b' = 4\lambda^2(2\lambda^2\pm1)-1$.

The series can be formed by finding the values of λ which admit of $2\lambda^2\pm1 = \square$. They are 1, 2, 5, 12, 29, 70, 169, 408, 985, ..., where $\lambda_n = 2\lambda_{n-1}+\lambda_{n-2}$.

The corresponding values are as follows:—

λ	$(2\lambda^2\pm1)^{\frac{1}{2}}$	a	b	c	b'	a'
1	1	3	4	5	3	4
2	3	17	144	145	143	24
5	7	99	4900	4901	4899	140
12	17	577	166464	166465	166463	816
29	41	3363	5654884	5654885	5654883	4756
...	...	...	...	...	...	...

It will be seen that $(2\lambda^2\pm1)^{\frac{1}{2}}$ follows the same law as λ, namely, that its n-th value is twice the $(n-1)$-th plus the $(n-2)$-th. It may also be observed that the values of the sides bear out the statement made in Question 15360 (*Educational Times* for June, 1903) as to right-angled triangles.

(III.) Here $c^2 = a^2+b^2 = a'^2+b'^2$. Hence c, a, b, a', b' must be of the following forms:—$c = t^2+u^2$ and also $c = t'^2+u'^2$; $a = t^2-u^2$,

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$b = 2tu$; $a' = 2t'u'$, $b' = t'^2 - u'^2$. Hence

$$c - b = (t^2 + u^2) - 2tu = 1, \quad b - b' = 2tu - (t'^2 - u'^2) = 1 \ldots (1, 2),$$

$$c = t^2 + u^2 = t'^2 + u'^2 \ldots (3).$$

By (1), $$t - u = 1 \ldots (4).$$

By (2), (3), $(t-u)^2 - u'^2 = u'^2 - 1$; whence, by (4),

$$u' = \pm 1 \ldots (5).$$

Also, by (2), (3), $$(t+u)^2 - 2t'^2 = 1 \ldots (6).$$

The three results (4), (5), (6) suffice. An infinite number of values of $(t+u)$, t' may be found satisfying (6); after which equation (4) determines t, u; and $u' = 1$ always.

Two complete solutions are shown below;—

$t+u$	t'	u'	t	u	c	a	b	a'	b'
17	12	1	9	8	145	17	144	24	143
99	70	1	50	49	4901	99	4900	140	4899

The solutions rise rapidly in magnitude, the next being $c = 166465$.

9836. (Professor Lévy.)—Soient PA, PB, PC, PD les quatre normales issues d'un point quelconque P à une ellipse. Démontrer (1) que si la droite AB tourne autour d'un point fixe M, la corde CD enveloppe une parabole tangente aux deux axes; (2) la réciproque; et (3) trouver le lieu des foyers de ces paraboles lorsque le point M parcourt une droite donnée.

Solution by A. M. Nesbitt, M.A., Frances E. Cave, *and others.*

(1) If the chords $$lx + my = 1, \quad l'x + m'y = 1 \ldots (1, 2)$$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$ be such that the normals at their extremities are concurrent, then (C. Smith, § 196)

$$a^2ll' = b^2mm' = -1 \ldots (3).$$

Now, if (1) passes through (p, q), we find at once by ordinary methods that the envelope of (2) is

$$(b^2xp - a^2yq)^2 + 2a^2b^2(b^2ap + a^2yq) + a^4b^4 = 0 \ldots (4),$$

which is plainly a parabola touching the principal axes.

(2) Again, if (2) be a tangent to $\sqrt{(\lambda x)} + \sqrt{(\mu y)} + \sqrt{\nu} = 0$, we have $\lambda/l' + \mu/m' = \nu$, which, by virtue of (3), gives $a^2\lambda l + b^2m\mu + \nu = 0$, showing that (1) passes through the point $(a^2\lambda, b^2\mu)$.

Now we readily find for the focus of (4)

$$x/b^2p = y/a^2q = -a^2b^2/(b^4p^2 + a^4q^2) = \sqrt{(x^2+y^2)}/\sqrt{(b^4p^2 + a^4q^2)},$$

whence $b^4p^2 + a^4q^2 = a^4b^4/(x^2+y^2)$. Thus

$$p = -b^2x/(x^2+y^2), \quad q = -a^2y/(x^2+y^2);$$

so that, if (p, q) move on the line $\alpha x + \beta y = 1$, the locus of the focus is the circle $x^2 + y^2 + ab^2x + \beta a^2y = 0$.

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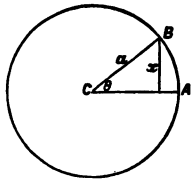
15391. (R. Chartres.) — Find the mean value of Z^n, where (1) Z = area of the triangle formed by joining two random points in a circle with the centre, (2) Z = volume of the tetrahedron formed by joining three random points in a sphere with the centre.

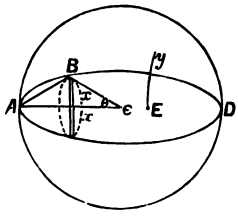
Solution by the Proposer.



(1) $M(Z^n) = (\frac{1}{2}a)^n M(x^n)[2/(n+2)]^2$ for A and B to move internally

$$= (\tfrac{1}{2}a^2)^n \frac{2^3}{(n+2)^2\pi} \int_0^{\frac{1}{2}\pi} \sin^n\theta\, d\theta.$$

(2) Let A and B be random points on the sphere, and y a perpendicular from any point in the solid hemisphere on the plane ABDC.



$$M(Z^n) = (\tfrac{1}{2}a)^n M(x^n)(\tfrac{1}{3})^n M(y^n) \left(\frac{3}{n+3}\right)^2$$

$$= (\tfrac{1}{2}a)^n a^n \int_0^{\frac{1}{2}\pi} \sin^{n+1}\theta\, d\theta\, (\tfrac{1}{3})^n \times \frac{3a^n}{(n+1)(n+3)} \left(\frac{3}{n+3}\right)^2$$

$$= (\tfrac{1}{6}a^3)^n \frac{3^3}{(n+1)(n+3)^3} \int_0^{\frac{1}{2}\pi} \sin^{n+1}\theta\, d\theta.$$

If $n = 1$, then $M_1 = \frac{9}{1024}\pi a^3$.

15344. (R. W. D. Christie.)—Prove that any integer

$$N \equiv r^n . d_n + r^{n-1} . d_{n-1} + \ldots + s,$$

when (1) r = any radix whatever, (2) s = sum of the digits, (3) d_n = n-th differential divided by $n!$ *Ex.:*

(1) $12345 \equiv 9^4 . 1 + 9^3 . 6 + 9^2 . 15 + 9^1 . 20 + 15 = 12345,$

where $r = 10,\ s = 15,\ d_4 = 1,\ d_3 = 6,\ d_2 = 15,\ d_1 = 20;$

(2) $54321 \equiv 7^5 . 1 + 7^4 . 10 + 7^3 . 32 + 7^2 . 46 + 7^1 . 37 + 15 = 54321,$

where $r = 8,\ s = 15,\ d_5 = 1,\ d_4 = 10,\ d_3 = 32,\ d_2 = 46,\ d_1 = 37.$

Solution by the Proposer.

(1) In Question 15191 (for solution of which see *Mathematical Reprint*, New Series, Vol. IV., p. 58) let $N = AX^2 + BXY + CY^2$; then

$du/dX = 2AX + BY$ and $du/dY = 2CY + BX,$

$du/dx = 2A(sx+1) + B(sy+1)$ and $du/dy = 2C(sy+1) + B(sx+1)$;

therefore

$$(2AX+BY)\{B(sx+1)+2C(sy+1)\} - (2CY+B)\{2A(sx+1)+B(sy+1)\} = 0.$$

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Thus $(B^2-4AC)(Y)(sx+1) = (B^2-4AC)(X)(sy+1)$; therefore

$$Y-X = s(Xy-Yx) = 0,$$

which proves the theorem. (See Prof. Young's *Calculus*, p. 82.) Similarly for higher forms.

(2) In Question 15191 we have $N = AX^n + BX^{n-1} + \ldots + Z$, where

$$X = (Ax^n + Bx^{n-1} + \ldots + Z)\,k + x = sk + x, \text{ say;}$$

then

$$N = S\left(\frac{k^n s^{n-1}}{n!}\frac{d^n u}{ds^n} + \frac{k^{n-1}s^{n-2}}{n-1!}\frac{d^{n-1}u}{ds^{n-1}} + \ldots + 1\right).$$

Now Question 15344 is a special case of this theorem where $x = 1$, X = any radix, or $r = sk + x = sk + 1$; therefore $k = 1/s(r-1)$. Thus

$$(k^n s^n/n!)(d^n u/ds^n)$$

becomes $\{(r-1)^n/n!\}\,d_n$, and generally

$$N = \{(r-1)^n/n!\}\,d_n + \{(r-1)^{n-1}/(n-1)!\}\,d_{n-1} + \ldots + S.$$

In the examples given we have

12345

$$= 1.10^4 + 2.10^3 + 3.10^2 + 4.10^1 + 5 = 9^4.d_4 + 9^3.d_3 + 9^2.d_2 + 9^1.d_1 + 15$$

where $d_1N\,(r=1)$ gives $4+6+6+4=20$, $(d_2/2!)N$ gives $(12+12+6)/2=15$, $\{(d_3)/3!\}\,N$ gives $(24+12)/6 = 6$, $(d_4/4!)\,N$ gives $1 = 1$. Hence

$$12345 = 9^4.1 + 9^3.6 + 9^2.15 + 9^1.20 + 15$$

and

$$54321 = 9^4.5 + 9^3.24 + 9^2.45 + 9^1.40 + 15.$$

It is clear the theorem leads to other developments.

15450. (Professor Langhorne Orchard, M.A., B.Sc.)—Find the product of the sum and the difference of the series—

$1 + 6 + 21 + 52 + 105 + \ldots$ to n terms, $3 + 14 + 39 + 84 + 155 + \ldots$ to n terms.

Solutions (I.) *by* Rev. T. Wiggins, B.A.; (II.) *by* C. M. Ross.

(I.) Let the sums and the terms of the series be respectively A, B; $u_1 + u_2 + \ldots$, $v_1 + v_2 + v_3 + \ldots$. Then $v_r + u_r = 2r(r^2+1)$; therefore

$$A + B = 2\left[\left\{\tfrac{1}{2}[n(n+1)]\right\}^2 + \tfrac{1}{2}[n(n+1)]\right] = \tfrac{1}{2}n(n+1)(n^2+n+2).$$

Also $v_r - u_r = 2r^2$; therefore $A - B = \frac{1}{3}[n(n+1)(2n+1)]$; whence

$$(A+B)(A-B) = \tfrac{1}{6}[n(n+1)]^2(2n+1)(n^2+n+2).$$

(II.) Sum of series $= 4 + 20 + 60 + 136 + 260$ ..... (i.),

difference of series $= 2 + 8 + 18 + 32 + 50$ ..... (ii.).

In (i.) the successive order of differences are

$$\begin{array}{cccc} 16 & 40 & 76 & 124 \\ & 24 & 36 & 48 \\ & & 12 & 12. \end{array}$$

Hence the n-th term is

$$4 + 16(n-1) + 12(n-1)(n-2) + 2(n-1)(n-2)(n-3) = 2n^3 + 2n$$

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In (ii.) the successive orders of differences are

$$6 \quad 10 \quad 14 \quad 18$$
$$4 \quad 4 \quad 4.$$

Hence the n-th term is $2+6(n-1)+2(n-1)(n-2) = 2n^2$;
therefore the sum of the series (i.)

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{2n(n+1)}{2} = \tfrac{1}{2}n(n+1)(n^2+n+2)$$

and the sum of the series (ii.) $= \frac{1}{3}n(n+1)(2n+1)$;

therefore product of (i.) and (ii.) $= \frac{1}{6}n^2(n+1)^2(2n+1)(n^2+n+2)$.

15249. (R. W. D. Christie.)—If r be any primitive root of a prime of the form $4m-1$, then three others are r^{-1}, $(ri)^2$, $(ri)^{-2}$, where $i = \sqrt{(-1)}$.

Solution by Lt.-Col. Allan Cunningham, R.E.

Let $\rho = 1/r$; then $\rho r = 1$; therefore $\rho^{\frac{1}{2}(p-1)} r^{\frac{1}{2}(p-1)} = 1$. But

$$r^{\frac{1}{2}(p-1)} \equiv -1 \pmod{p} \quad \text{[by hypothesis]};$$

therefore $\rho^{\frac{1}{2}(p-1)} \equiv -1 \pmod{p}$, and, since no exponent less than $\frac{1}{2}(p-1)$ will suffice to remove r from the above equality, therefore $\frac{1}{2}(p-1)$ is the lowest exponent for ρ also; so that ρ is a primitive root of p.

Next $(i^2)^{\frac{1}{2}(p-1)} = (-1)^{2m-1} = -1$; therefore

$$(r^2i^2)^{\frac{1}{2}(p-1)} \equiv (r^{p-1})(i^2)^{\frac{1}{2}(p-1)} \equiv (+1)(-1) \equiv -1 \pmod{p}.$$

Hence, as no exponent less than $\frac{1}{2}(p-1)$ will suffice to reduce r^2i^2 to -1, it follows that r^2i^2 is a primitive root. Hence (by the first theorem above) its reciprocal $(r^2i^2)^{-1}$ is also a primitive root.

15425. (I. Arnold.)—Two uniform beams of lengths 10 feet and 8 feet, and weights 100 lb. and 80 lb. respectively, are hinged at C and laid in a plane perpendicular to the axis across a smooth horizontal cylinder of 6 feet radius: it is required to find the position of equilibrium of the beams AC, BC; also the strain on the hinge and the direction in which it acts.

Solution by the Proposer.

Let STL be a section of the cylinder perpendicular to its axis; AC, BC the beams hinged at C; S, T tangential points; g, G the mid-points of the beams; O the centre and OF perpendicular to the horizontal plane

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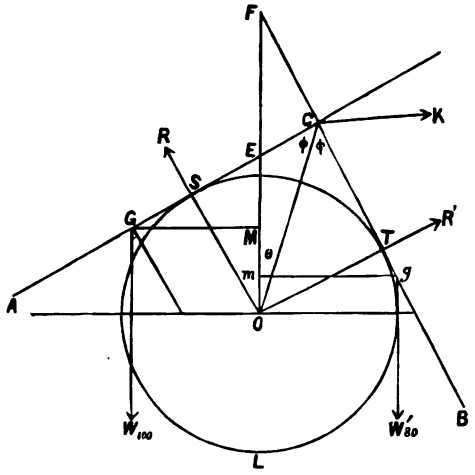
and meeting BC produced in F. Draw GM, gm perpendicular to OF, and join OC. Let $\phi = \angle ACO$ or $\angle BCO$ and $\theta = \angle FOC$.



Now, if $\bar{z}$ be the height of the common centre of gravity of the beams above the horizontal plane through O, then $(W + W')\bar{z} = W.OM + W'.Om$. Now, since OF cuts AC in E,

$$OM = OC\cos\theta - CG\cos CEF = \gamma(\cos\theta/\sin\phi) - a\cos(\phi+\theta)$$

and $$Om = OC\cos\theta - Cg\cos F = \gamma(\cos\theta/\sin\phi) - b\cos(\phi-\theta).$$

For facility take $AC = 2a = 10$, $BC = 2b = 8$, $OS = \gamma = 6$, $W = 100$, and $W' = 80$; therefore

$$(W + W')\bar{z} = (W + W')\gamma(\cos\theta/\sin\phi) - Wa\cos(\phi+\theta) - W'b\cos(\phi-\theta).$$

Now, in a position of equilibrium $\bar{z}$, which is a function of ϕ and θ, must be a maximum or minimum. Consequently, differentiating first with respect to θ and then ϕ, and dividing by differentials, we get

$$0 = (W + W')d_\theta\bar{z} = -(W + W')\gamma\frac{\sin\phi}{\sin\theta} + Wa\sin(\phi+\theta) - W'b\sin(\phi-\theta)$$

and ..... (1)

$$0 = (W + W')d_\phi\bar{z}$$

$$= -(W + W')\gamma\frac{\cos\theta\cos\phi}{\sin^2\phi} + Wa\sin(\phi+\theta) + W'b\sin(\phi-\theta) \text{ .....} (2).$$

Performing the operations indicated in these equations and substituting $\operatorname{cosec}^2\phi$ for $1/(\sin^2\phi)$, we get

$$(W + W')\gamma\operatorname{cosec}^2\phi = (Wa - W'b)\cot\theta + (Wa + W'b)\cot\phi \text{ .....} (3)$$

and $$(W + W')\gamma\operatorname{cosec}^2\phi = (Wa - W'b)\tan\theta + (Wa + W'b)\tan\phi \text{ ...} (4).$$

Taking the difference of these (3) and (4), transposing, &c., we get

$$\frac{W'b - Wa}{Wa + W'b} = \frac{\cot\phi - \tan\phi}{\cot\theta - \tan\theta} = \frac{2\cot 2\phi}{2\cot 2\theta} = \frac{\tan 2\theta}{\tan 2\phi} = -\frac{9}{41} \text{ .....} (5).$$

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In order to eliminate θ, transpose equations (3) and (4), and they become

$$(Wa-W'b)\cot\theta = (Wa+W')\gamma\operatorname{cosec}^2\phi-(Wa+W'b)\cot\phi \quad \ldots\ldots (6)$$

and $(Wa-W'b)\tan\theta = (Wa+W')\gamma\operatorname{cosec}^2\phi-(Wa+W'b)\tan\phi \quad \ldots (7).$

Now take the product of (6) and (7) and

$$(Wa-W'b)^2 = (W+W')^2\gamma^2\operatorname{cosec}^4\phi-(W+W')\gamma\operatorname{cosec}^2\phi\tan\phi(Wa+W'b)$$
$$-(W+W')\gamma\operatorname{cosec}^2\phi\cot\phi(Wa+W'b)+(Wa+W'b)^2 \ldots (8).$$

Transposing and substituting $1/(\sin^2\phi)$ for $\operatorname{cosec}^2\phi$ and $1/(\sin^4\phi)$ for $\operatorname{cosec}^4\phi$,

$$\frac{(W+W')\gamma}{\sin^2\phi}\left[\frac{\tan^2\phi+1}{\tan\phi}\right][Wa+W'b] = \frac{(W+W')^2}{\sin^4\phi}+4WW'ab \ldots (9),$$

or

$$4WW'ab\sin^4\phi+(W+W')^2\gamma^2$$
$$=(W+W')(Wa+W'b)\gamma\sin^2\phi+\left(\frac{\tan^2\phi+1}{\tan\phi}\right) \quad \ldots\ldots (10).$$

But $\sin^2\phi[(\tan^2\phi+1)/\tan\phi] = \tan\phi$; therefore

$$4WW'ab\sin^4\phi-(W+W')(Wa+W'b)\gamma\tan\phi+(W+W')^2\gamma^2 = 0 \ldots (11).$$

By substituting the values of a, b, W, W', and γ we get

$$64000\sin^4\phi-885600\tan\phi+1166400 = 0 \ldots\ldots\ldots\ldots (12).$$

Dividing by 640,000,

$$\sin^4\phi-1{\cdot}38375\tan\phi+1{\cdot}8225 = 0 \ldots\ldots\ldots\ldots (13).$$

By approximation ϕ becomes $59^\circ 50'$ very nearly.

Taking $\phi = 59^\circ 50'$ in equation (5), we get $\theta = 10^\circ 32'$. Hence the position of equilibrium is obtained.

The force exerted upon AC in the direction CK and upon BC in the opposite direction may be obtained by resolving the forces which keep AC at rest parallel to AC (to avoid the reaction at S) and taking moments about S; then

$$0 = K\cos(\pi-ACK)-W.SG\cos AEO$$

and $$0 = KSC\sin ACK-W.SG\sin AEO.$$

From these equations we find

$$(K/W)^2$$
$$=\cos^2 AEO+SG^2/SC^2\sin^2 AEO = \cos^2(\phi+\theta)+(a/\gamma\tan\phi-1)^2\sin^2(\phi+\theta);$$

or, substituting the numbers,

$$\tfrac{1}{10000}K^2 = \cos^2 70^\circ 22'+(\tfrac{4}{5}\tan 59^\circ 50'-1)^2\sin^2 70^\circ 22' = {\cdot}279779,$$

or $K^2 = 2797{\cdot}79$; therefore $K = 52{\cdot}89$ lb. force on AC. But

$$\tan ACK = -SG/SC\tan AEO = (1-\tfrac{4}{5}\tan 59^\circ 50')\tan 70^\circ 22'$$
$$=-1{\cdot}214 = \tan(180^\circ-50^\circ 30') = 129^\circ 30' = ACK,$$

showing the direction in which the force CK acts. When the beams are equal in length and weight then θ vanishes and equation (4) becomes

$$2W\gamma\operatorname{cosec}^3\phi = 2Wa\tan\phi \quad \text{or} \quad \gamma\operatorname{cosec}^2\phi = a\tan\phi;$$

or $$\gamma/\sin^2\phi = a\sin\phi/\cos\phi \quad \text{or} \quad \gamma\cos\phi = a\sin^3\phi;$$

and, if B is the angle made by the beams with the horizon, then

$$\gamma\sin B = a\cos^3 B$$

(see W. S. Aldis, *Rigid Dynamics*, p. 100, Art. 118).

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9707. (Professor KALIPADA BASU.)—Evaluate

$$\int \frac{dx}{(a\cos^2 x + b\sin^2 x)^2} \quad \text{and} \quad \int \frac{dx}{(a\cos^2 x + b\sin^2 x)^3}$$

by differentiating a certain easy integral.

Solution by H. J. WOODALL, A.R.C.S.

We have $\int \frac{dx}{a\cos^2 x + b\sin^2 x} = \frac{1}{\sqrt{(ab)}} \tan^{-1}\left(\sqrt{\frac{b}{a}}\tan x\right)$.

Differentiate with respect to a; therefore

$$\int \frac{\cos^2 x\, dx}{(a\cos^2 x + b\sin^2 x)^2} = \frac{1}{2a\sqrt{(ab)}} \tan^{-1}\left(\sqrt{\frac{b}{a}}\tan x\right) + \frac{1}{2a}\frac{\tan x}{a + b\tan^2 x}.$$

Similarly, by differentiation with respect to b,

$$\int \frac{\sin^2 x\, dx}{(a\cos^2 x + b\sin^2 x)^2} = \frac{1}{2b\sqrt{(ab)}} \tan^{-1}\left(\sqrt{\frac{b}{a}}\tan x\right) - \frac{1}{2b}\frac{\tan x}{a + b\tan^2 x}.$$

By addition we get the required result

$$\int \frac{dx}{(a\cos^2 x + b\sin^2 x)^2} = \frac{a+b}{2\,(ab)^{\frac{1}{2}}} \tan^{-1}\left(\sqrt{\frac{b}{a}}\tan x\right) - \frac{a-b}{2ab}\frac{\tan x}{a + b\tan^2 x}.$$

Differentiate this with respect to a and b separately, and add the results; therefore

$$\int \frac{dx}{(a\cos^2 x + b\sin^2 x)^3} = \frac{3a^2 + 2ab + 3b^2}{8\,(ab)^{\frac{1}{2}}} \tan^{-1}\left(\frac{b}{a}\tan x\right) - \frac{(a-b)\tan x\,(3a^2 + 5ab\sec^2 x + 3b^2\tan^2 x)}{8a^2b^2\,(a + b\tan^2 x)^2}.$$

15459. (C. E. YOUNGMAN, M.A.)—Three fixed circles S_1, S_2, S_3 touch two by two at A, B, C; and two variable circles V, V′, touching each other at P, are drawn so that V touches S_1 and S_2 and V′ touches S_1 and S_3. Show that the locus of P is the inverse of an ellipse, its trilinear equation in ABC being $8\,(\Sigma\beta\gamma\sin A)^2 = (Y + 3Z - X)(3Y + Z - X)$, where X, Y, Z stand for $\beta^2 + 2\beta\gamma\cos A + \gamma^2$, ...

Solution by R. F. DAVIS, M.A.

Let s_1, s_2 be fixed parallel tangents to a given circle s_3 (centre o, radius r), the respective points of contact being b, a. Let v be the centre of a variable circle touching s_1 and s_2 in m, n respectively, and therefore of radius r; v' the centre of another variable circle touching s_1 and s_3 in q, t respectively of radius ρ; and suppose the circles of which v and v' are the centres touch each other at p. Then, if $ov = h$, $\frac{1}{4}h^2 = bq^2 = mq^2 = 4r\rho$. Thus npq intersects ab produced in a point e such that $ab = be = 2r$ and $oe = 3r$. If x, y are the co-ordinates of p referred to oe, ov as axes, then

$$y^2 : (3r-x)^2 = h^2 : 16r^2 = \rho : r = (r-x) : (r+x)$$

or $y^2 = (3r-x)^2(r-x)/(r+x)$, which is the equation to the inverse of an

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ellipse. Transferring origin to e, it becomes $x(x^2+y^2) = 2r(2y^2+x^2)$. The equation may also be written

$$32r^3y^2 = [3(pb)^2+4r^2-(pa)^2][(pb)^2+12r^2-(pa)^2]$$

or again $(x+r)(x^2+y^2-r^2) = 8r(r-x)^2$.

The theorem of the Question may now be derived by inverting according to well known rules with respect to any point C. One form of the locus of P is $S_2 \cdot S_3 = \kappa S_1^2$ where $\kappa^2 = 4c_2c_3/c_1^2$, c_1, c_2, c_3 being the radii.

The Proposer contributes the following discussion: — To get the ellipse, draw the circle ABC and its symmedian chord AA′ passing through the centre of S_1, and invert from A′. Then A, B, C become collinear points a, b, c, and $ba = ca$. In the new figure let v touch s_1, s_2, v' at m, n, p; and v' touch s_1, s_3 at m', n'. It is easily shown that the circles $bm'n'$ and cmn touch bc; that $bn'pm$ and $cnpm'$ are straight lines; and that mn, $m'n'$ are parallel and trisect bc. Then, writing θ, ϕ for the angles pbc, pcb, we shall get $\phi = cmn$ and $\cot\theta \cot\phi = 2:1$, the ratio in which mn divides bc. This shows that p describes an ellipse on bc as axis major, with eccentricity $\frac{1}{3}\sqrt{2}$; and that in the original figure P is given by the intersection of circles BA′, CA′ which cut ABCA′ at angles θ, ϕ such that $\cot\theta \cot\phi = 2$. Now the equation of ABCA′ is $\Sigma\beta\gamma \sin A = 0$; of the circle BA′ orthogonal to ABCA′ it is $3Y+Z-X = 0$; and of the required circle BA′ it is $2\cot\theta \cdot \Sigma + (3Y+Z-X) = 0$; and the equation in the Question follows at once.

15445. (Lt.-Col. Allan Cunningham, R.E.)—*Analogue of Fermat's Theorem.*—Let $F_n = 2^{2^n}+1$, a Fermat's number, and $P = F_a \cdot F_b \cdot F_c \ldots F_z$. Then $2^{P-1} \equiv +1 \pmod P$ and $2^{P(P-1)} \equiv +1 \pmod{P^2}$ for all values of P when $a<b<c<\ldots<z$, $a \not< 2$, and $z \not> 2^a-1$.

Solution by the Proposer.

For shortness let $2^x = X$, $2^z = Z$, $x<z$; ϵ = an even number.

(i.) $F_x = 2^X+1$; $2^{(F_x-1)} = 2^{2^X} = (2^X)^{1/X \cdot 2^X}$; but $2^X \equiv -1 \pmod{F_x}$, and $1/X \cdot 2^X$ is always even; therefore

$$2^{(F_x-1)} \equiv (-1)^\epsilon \equiv +1 \pmod{F_x} \text{ always } \ldots\ldots\ldots\ldots (1);$$

(ii.) $2^Z \equiv -1 \pmod{F_z}$; $2^{2^X} = (2^Z)^{1/Z \cdot 2^X} \equiv (-1)^{1/Z \cdot 2^X} \equiv +1 \pmod{F_z}$, provided $1/Z \cdot 2^X = \epsilon$. This involves $Z \not> \frac{1}{2} \cdot 2^X$, or $2^z \not> \frac{1}{2} \cdot 2^{2^x} = 2^{(2^x-1)}$, or $z \not> (2^x-1)$; and, as $x<z$, this requires also $x \not< 2$. Hence, with these conditions, $2^{2^x} \equiv +1 \pmod{F_x \text{ and } F_z} \equiv +1 \pmod{F_x \cdot F_z}$. Now $F^x \cdot F^z$

$= (2^X+1)(2^Z+1) = 2^{X+Z}+2^Z+2^X+1 = 2^X(2^Z+2^{Z-X}+1)+1 = 2^X \cdot M+1$, suppose; therefore

$$2^{F_x \cdot F_z - 1} = 2^{2^X \cdot M} \equiv +1 \pmod{F_x \cdot F_z} \ldots\ldots\ldots\ldots (2).$$

(iii.) By reasoning as in (ii.) result (2) may be extended to the product $P = F_a \cdot F_b \ldots F_z$, with the same limits as in (ii.), and as in the Question. Therefore $2^{P-1} \equiv +1 \pmod P$, as required.

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(iv.) By a similar set of steps it may be shown that

(i.a) $2^{(F_x-1)F_x} \equiv +1 \pmod{F_x^2}$, (ii.$a$) $2^{(F_x \cdot F_x-1)F_xF_x} \equiv +1 \pmod{F_x^2 F_x^2}$;

hence (iii.a) $2^{(P-1)P} \equiv +1 \pmod{P^2}$.

15467. (Professor M. W. Crofton, F.R.S.)—A floor is ruled with parallel lines 1 foot asunder. If an equilateral triangle whose side is 1 foot be thrown on the floor, show that the chance of it falling on one of the lines is $3/\pi$.

Solution by D. Biddle.

As stated in the solution to Question 15130 (see *Reprint*, Vol. III., New Series, pp. 47, 48), the following law was enunciated (*not* for the first time) on p. 98, Vol. LII., under Question 10219 (Prof. Sylvester):—"The ratio borne to the periphery of the given circle (touching two adjacent parallels) by the length of a string drawn tightly round any plane figure whatsoever (capable of lying in all positions between the parallels) is invariably the probability that such figure will be cut by a parallel if disposed at random on a ruled floor." Hence, &c. [Proof of the above theorem is given in Vol. LII., at the place referred to.]

15418. (W. J. Greenstreet, M.A.)—The foci of a variable ellipse are fixed. From a point P normals are drawn to the variable ellipse. Find the loci of the centre and foci of the conic passing through the point P and through the feet of the normals.

Solutions (I.) *by* R. F. Davis, M.A., *and* H. Bateman, B.A.; (II.) *by the* Proposer.

(I.) Draw PN, PM perpendicular to CA, CB respectively. Produce PN to Q and MP to R so that $PN = e^2PQ$ and $MP = e^2MR$; and complete the rectangle RPQO. Then it is known that a rectangular hyperbola having O for its centre, OQ, OR for its asymptotes, and passing through C, P will intersect the ellipse in four points, the normals at which co-intersect at P. [See solution, by R. F. Davis, M.A., to Question 52, *Math. Gaz.*, Vol. I., p. 21.] Then, since PN : RO = MP : MR, the locus of O is the straight line through M, N. Moreover, if F be a focus of the rectangular hyperbola, FP ~ FC varies as the difference of the perpendicular distances of P, C from the directrix, that is to say, is constant. Hence the locus of F is a hyperbola having P, C for foci (passing through M, N).

It is to be noticed that these results do not involve the lengths of the axes. The restriction that the variable ellipses should be confocal may therefore be removed. It will be sufficient for them to have the same centre and their axes in the same direction.

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(II.) If P be (ξ, η), and the equation to the ellipse be $x^2/a^2 + y^2/b^2 = 1$, that of the Apollonian hyperbola is $c^2xy + b^2\eta x - a^2\xi y = 0$. The four foci are found to be $[a^2\xi \pm ab\sqrt{(\pm 2\xi\eta)}]/c^2$, $-[b^2\eta \pm ab\sqrt{(\pm 2\xi\eta)}]/c^2$, and the locus of the foci consists of two conics, whose equations are

$$(\xi \pm \eta)^2 (x^2 + y^2) = (\xi x + \eta y \pm \xi\eta)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

If O be the centre of the original ellipse, it is also a focus of each of these conics. Their centre will be found to bisect OP. O and P are common foci. The locus of the centre is $x/\xi + y/\eta - 1 = 0$, *i.e.*, the join of the projections of P on the axes. As the three loci are independent of a and b, the ellipses need not be confocal. [The conics (A) will always be an ellipse and a hyperbola.]

Mr. Bateman also remarks that the locus of the foci consists of two conics, and Mr. Davis draws attention to the fact that the existence of the second of the conics becomes evident in analysis from the consideration of the imaginary foci, of which pure geometry does not take account.

The Four Simson-Lines of Four Points on a Circle, each with regard to the Triangle formed by the other three, are Concurrent.

By W. F. Beard, M.A.

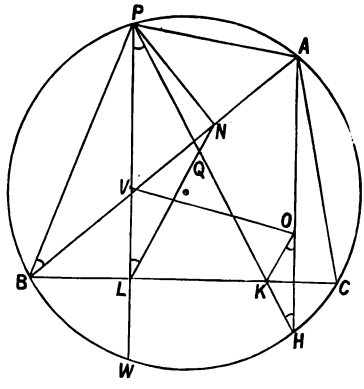
If O be the orthocentre of ABC, and PL, PN perpendiculars from P to BC and AB, then LN is the Simson-line of P with respect to ABC. Now plainly the following angles are equal :—

$$\angle KOH = \angle KHO = \angle LPK = \angle PBA = \angle PLQ.$$

Since $\angle LPK = \angle PLQ$ in the right-angled triangle PLK, we get PQ = QK ; while, since

$$\angle KOH = \angle PLQ,$$

OK is parallel to LN ; therefore PO is bisected by LN.



Again, taking LV = LW, V is the orthocentre of PBC, and, PW, AH being parallel chords, PA = WH = OV (O and V being the images of H and W in BC). Hence OV is equal and parallel to AP ; therefore APVO is a parallelogram, and AV is bisected at the middle point of OP. But the Simson-line of A with regard to PBC bisects AV (by the preceding theorem) ;

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therefore it also bisects OP. Hence the four Simson-lines are concurrent at the middle point of OP.

The further theorem is likewise obvious: that the four orthocentres of ABC, PBC, PCA, PAB form a quadrilateral equal in all respects to PABC.

15429. (H. MacColl, B.A.)—What is the chance that xy is greater than a: firstly, on the assumption that x and y are each taken at random between 1 and 5; secondly, on the assumption that x is taken at random between 1 and 5, and then y at random between 1 and x? The constant a is positive.

Solution by the Proposer.

We will first find the limits of variation on each assumption. Let A = our data on the first assumption, and let Q = $(xy > a)$. We have

$$A = y \quad x_{1'.2}, \quad Q = y_3;$$

therefore

$$AQ = y_{1'.2.3}\, x_{1'.2}$$
$$= y_{1'}\,(y_2 x_2 + y_3 x_{3'})\, x_{1'.2}$$
$$= y_{1'.2} x_{1'.2.3} + y_{1'.3} x_{3'.1'.2}$$
$$= y_{1'.2} x_{1'.2.3} + y_{1'.3} x_{3'.1'.2.4} \text{ [for } y_{1'.3} \text{ implies } y_1 > y_3\text{, which implies } x_4\text{]}$$
$$= y_{1'.2} x_{1'}\,(x_2 a_{1'} + x_3 a_1) + y_{1'.3}\,(x_{3'} a_{2'} + x_{1'} a_2)(x_2 a_{2'} + x_4 a_2)$$
$$= y_{1'.2}\,(x_{1'.2} a_{1'} + x_{2'.3}\, a_1) + y_{1'.3}\,(x_{3'.2}\, a_{2'} + x_{1'.4}\, a_2)$$

[for $a_{2'.2}$ is impossible]

$$= y_{1'.2}\,(x_{1'.2}\, a_{1'.0} + x_{1'.3} a_{2'.1}) + y_{1'.3}\,(x_{3'.2}\, a_{2'.1} + x_{1'.4}\, a_{3'.2})$$

[assuming a to be positive; for $x_{1'.3} : (x_1 > x_3) : a_{2'}$, $x_{3'.2} : a_1$, and $x_{1'.4} : a_{3'}$].

Limits on First Assumption.		
$y_1 = 5$	$x_1 = 5$	$a_1 = 1$
$y_2 = 1$	$x_2 = 1$	$a_2 = 5$
$y_3 = a/x$	$x_3 = a$	$a_3 = 25$
	$x_4 = \frac{1}{5}a$	

Arranging according to the limits of the constant, we get finally on the first assumption

$$a_{1'.0}\,(y_{1'.2}\, x_{1'.2}) + a_{2'.1}\,(y_{1'.2} x_{1'.3} + y_{1'.3} x_{3'.2}) + a_{3'.2}\,(y_{1'.3} x_{1'.4}).$$

To find the limits on the second assumption, let A = our data $= y_{1'.2} x_{1'.2}$, and let

$$Q = (xy > a) = y_3.$$

$$AQ = y_{1'.2.3}\, x_{1'.2}$$
$$= y_{1'.2} x_{1'.2.3} + y_{1'.3} x_{3'.1'.2.4},$$

as before, but with different values of y_1 and x_4. By the same process as before we get finally

Limits on Second Assumption.		
$y_1 = x$	$x_1 = 5$	$a_1 = 1$
$y_2 = 1$	$x_2 = 1$	$a_2 = 5$
$y_3 = a/x$	$x_3 = a$	$a_3 = 25$
	$x_4 = \sqrt{a}$	

$$AQ = a_{1'.0}\,(y_{1'.2} x_{1'.2}) + a_{2'.1}\,(y_{1'.2} x_{1'.3} + y_{1'.3} x_{3'.4}) + a_{3'.2}\,(y_{1'.3} x_{1'.4}).$$

Thus, whichever of the two assumptions we take as our data, AQ is

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possible only in three cases, namely, $a_1 > a > 0$, $a_2 > a > a_1$, and $a_3 > a > a_2$. To find the probability of any statement $y_{m'.n}\, x_{r'.s}$ in any of the three cases, and on either assumption, let the symbol x^a assert as a statement that, out of the many possible and equally probable values of x between x_1 and x_2, a particular value x is the one that will turn up; and let y^a similarly assert that, out of the many possible and equally probable values of y between y_1 and y_2, a particular value y is the one that will turn up. We have

$$x^a y^a/\mathrm{A} = x^a/\mathrm{A} \,.\, y^a/\mathrm{A}x^a = dx/(x_1 - x_2) \,.\, dy/(y_1 - y_2),$$

and the sum of these infinitesimal chances within the limits prescribed by the statement $y_{m'.n}\, x_{r'.s}$ is the chance that this prescribing statement is true. Hence, whatever assumption we take,

$$\frac{y_{m'.n}\, x_{r'.s}}{\mathrm{A}} = \int_{y_n}^{y_m} dy \int_{x_s}^{x_r} dx \frac{1}{(x_1 - x_2)(y_1 - y_2)}.$$

Taking the first assumption, we have $(x_1 - x_2)(y_1 - y_2) = 16$; so that, on the first assumption, we get three different answers depending upon the limits of a, namely,

(1) $\mathrm{Q}/\mathrm{A}a_{1'.0} = 1$, (2) $\mathrm{Q}/\mathrm{A}a_{2'.1} = \frac{1}{16}(15 + a - a\log a)$,

(3) $\mathrm{Q}/\mathrm{A}a_{3'.2} = \frac{1}{16}(25 - a + a\log a - 2a\log 5)$.

When a is greater than a_3 the chance is, of course, zero. As a particular case, suppose $a = 4$. This is included in the second case, and we get $\mathrm{Q}/\mathrm{A} = \frac{1}{16}(15 + 4 - 4\log 4) = \cdot 8409$ nearly. Next, suppose $a = 9$. This is included in the third case, and we get

$$\mathrm{Q}/\mathrm{A} = \tfrac{1}{16}(25 - 9 + 9\log 9 - 18\log 5) = \cdot 4253 \text{ nearly.}$$

Taking now the second assumption, that y is taken at random between 1 and x, we have $(x_1 - x_2)(y_1 - y_2) = 4(x - 1)$, from the second table of limits, and again three answers, namely,

(1) $\mathrm{Q}/\mathrm{A}a_{1'.0} = 1$,

(2) $\mathrm{Q}/\mathrm{A}a_{2'.1} = \frac{1}{4}\left\{5 - \sqrt{a} + a\log\sqrt{a} - (a-1)\log(\sqrt{a}+1)\right\}$,

(3) $\mathrm{Q}/\mathrm{A}a_{3'.2} = \frac{1}{4}\left\{5 - \sqrt{a} + a\log 5 - (a-1)\log 4 - a\log\sqrt{a} + (a-1)\log(\sqrt{a}-1)\right\}$.

When a is greater than a_3 the chance is zero.

As particular cases, when $a = 4$, we get $\mathrm{Q}/\mathrm{A} = \cdot 6192$ nearly, and when $a = 9$, we get $\mathrm{Q}/\mathrm{A} = \cdot 2633$ nearly.

It is assumed throughout that a is positive; for on the assumption that a is negative there would be no problem to solve, the chance required in that case being evidently 1, since x and y are restricted between 1 and 5. (For a full explanation of the method see my paper on "The Calculus of Limits," *Reprint*, New Series, Vol. v.)

Note.—I may mention that I have tested the last two chances (·6192 and ·2633) by an easy experimental method of 500 trials, and found that the event $xy > 4$ happened 320 times instead of the most likely number, *a priori*, 310; while the event $xy > 9$ happened 137 times instead of the most likely number 132. These results are, I think, as close as could reasonably be expected.

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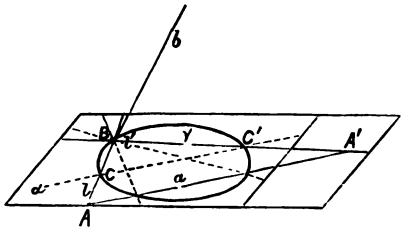
8712. (Professor SCHOUTE.)—A surface S of the third order passing through two non-coplanar lines a and b is given, and a line l intersects a in A, b in B, and S for the third time in C, under the condition AC = CB. Find (1) the locus of the line l, and (2) its intersection with the given surface S.

Solution by Professor JAN DE VRIES.

A plane α through a cuts S still in a conic γ passing by the point B where a cuts b. Hence there are two lines l through B, intersecting γ in C and a in A, under the condition AC = CB. Hence b is a double line of the locus (l), and so will be a. Hence the locus (l) is a ruled surface of the fourth order with two double lines a and b.



Its intersection with S consists of the double lines a and b, each being counted twice, the five straight lines of S which cut both a and b, and of the cubic curve that S has in common with the plane which bisects at right angles the common perpendiculars of a and b.

[Mr. Nesbitt's note (*Educational Times*, Vol. LVII., p. 40, and *Mathematical Reprint*, Vol. VI., New Series, p. 29) therefore loses significance.]

Note on Minimum Deviation through a Prism.

By R. CHARTRES.

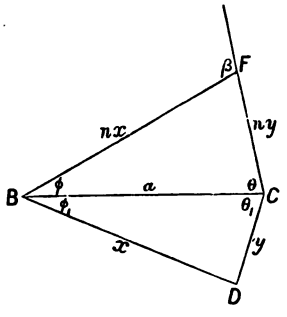
(By kind permission of the *Philosophical Magazine*.)

$\sin\theta = \mu \sin\theta_1$, $\sin\phi = \mu \sin\phi_1$;

$\theta + \phi$ to be a minimum;

$\theta_1 + \phi_1$ = constant.

$\sin\beta/\sin\phi = a/ny = \sin D/(n \sin\phi_1)$; therefore $n \sin\beta = \mu \sin D$ = constant. Now, if ϕ_1 or θ_1 be small, n will practically = unity, while at the symmetrical position $n^2 = (1 - \sin^2\phi_1)/(1 - \mu^2 \sin^2\phi_1)$; therefore n = maximum at the symmetrical position; therefore β = minimum; therefore deviation is a minimum there.



Or, without n, if ϕ_1 is small,

$$\phi - \phi_1 = (\mu - 1)\phi_1;$$

but $\theta - \theta_1 > (\mu - 1)\theta_1$, decreasing with θ_1. Now $\theta_1 + \phi_1$ = constant;

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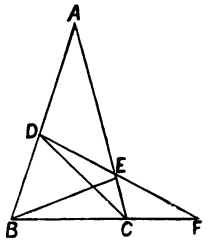
therefore the increment ϕ_1 = decrement of θ_1; therefore in the expression $(\theta-\theta_1)+(\phi-\phi_1)$ the increase in $(\phi-\phi_1)$ is less than the decrease in $(\theta-\theta_1)$ as ϕ_1 increases up to its value at the symmetrical position; therefore the expression or deviation is then a minimum.

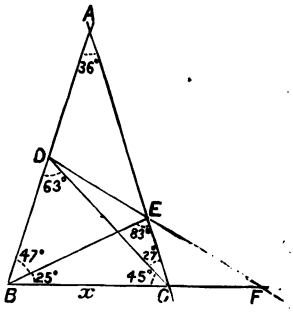
15474. (*Communicated by* ARTHUR ANSTEY, B.A., L.C.P.)—ABC is an isosceles triangle, base angles being 72°; $\angle EBC = 25°$, $\angle DCB = 45°$; DE is joined and produced to meet the base produced in F. Find the angle EFC.



Solutions (I.) *by* C. M. ROSS; (II.) *by* CONSTANCE I. MARKS, B.A.; (III.) *by* W. SCRIMGEOUR, M.A., B.Sc., *and* S. C. BAGCHI, B.A.

(I.) $$CD = \frac{x \sin 72°}{\sin 63°},$$

$$CE = \frac{x \sin 25°}{\sin 83°}.$$



Again $CED + CDE = \pi - 27°$,

$$\tan \tfrac{1}{2}(CED - CDE)$$

$$= \frac{CD - CE}{CD + CE} \cot \frac{27°}{2}$$

$$= \frac{\sin 72° \sin 83° - \sin 25° \sin 63°}{\sin 72° \sin 83° + \sin 25° \sin 63°} \cot 13\tfrac{1}{2}°$$

$$= \frac{\cos 11° + \cos 25° - \cos 38° + \cos 88°}{\cos 11° + \cos 25° + \cos 38° - \cos 88°}$$

$$= \frac{1{\cdot}1348237}{2{\cdot}6410463} \times 4{\cdot}1652998$$

$$= 1{\cdot}7892 \text{ (approximately)}$$

$$= \tan 60° 48';$$

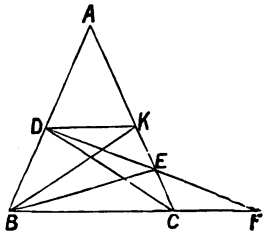
therefore $\frac{1}{2}(CED - CDE) = 60° 48'$, $\frac{1}{2}(CED + CDE) = 76° 30'$;

therefore $CDE = 15° 42'$;

therefore $EFC = \pi - 108° - 27° - 15° 42' = 29° 18'$ (approximately).

(II.) Draw DK parallel to BC, and join BK. Then the triangle BCK is easily shown to be equal to the triangle DBC in every respect. Hence

$$\angle KBE = 20°, \quad \angle BKC = 63°.$$



Also $\angle KCB = 72°$ (given);

and $\left.\begin{aligned} \angle DCE &= 27° \\ \angle DKC &= 108° \\ \angle BEC &= 83° \end{aligned}\right\}$ (evidently).

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By a well known theorem which we will call (X),

$$CK \cot 83° = CE \cot 63° - KE \cot 72°$$

and $$CK \cot 83° = KE \cot 20° - CE \cot 25°.$$

By subtraction, $0 = CE(\cot 63° + \cot 25°) - KE(\cot 72° + \cot 20°)$; therefore

$$\frac{CE}{KE} = \frac{\cot 72° + \cot 20°}{\cot 63° + \cot 25°} = \frac{\cdot3249197 + 2\cdot7474774}{\cdot5095254 + 2\cdot1445069} \text{ (from Chambers' Tables)}$$

$$= \frac{3\cdot0723971}{2\cdot6540323}.$$

Let $\theta = \angle DEK$; then, by (X), $CK \cot\theta = KE \cot 27° - CE \cot 108°$,

$$5\cdot7264294 \cot\theta = 2\cdot6540323 \cot 27° - 3\cdot0723971 \cot 108°;$$

therefore $$\cot\theta = \frac{2\cdot6540323 . 1\cdot9626105 + 3\cdot0723971 . \cdot3249197}{5\cdot7264294}$$

(from Chambers' Tables)

$$= 1\cdot0839414, \text{ approximately.}$$

Using the usual methods for proportional parts, we find $\theta = 42° 41' 36''$ nearly; therefore $\angle EFC = 72° - 42° 41' 36''$ nearly $= 29° 18' 24''$ nearly.

(III.) Let $BC = a$. Find EC and DC in terms of a, using natural sines to two places.

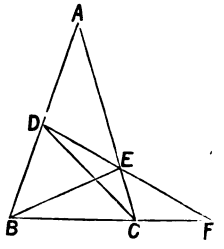
$\sin 25° = \cdot42$, $\sin 83° = \cdot99$, $\sin 72° = \cdot95$, $\sin 63° = \cdot89 = \cos 27°$ and $\sin 27° = \cdot45$, $\sin 15° = \cdot26$.

$$\frac{EC}{a} = \frac{\sin EBC}{\sin BEC} = \frac{\sin 25°}{\sin 83°};$$

therefore $$EC = \frac{a(\cdot42)}{\cdot99} = a(\cdot42).$$

$$\frac{DC}{a} = \frac{\sin DBC}{\sin BDC} = \frac{\sin 72°}{\sin 63°};$$

therefore $$DC = \frac{a(\cdot95)}{\cdot89} = a(1\cdot07).$$



And $DE^2 = DC^2 + CE^2 - 2DC . CE \cos DCE$;

therefore

$$DE^2 = a^2[(\cdot42)^2 + (1\cdot07)^2 - 2(\cdot42)(1\cdot07)(\cdot89)];$$

therefore $DE^2 = a^2(\cdot5165)$; therefore $DE = a(\cdot72)$.

$$\sin CDE = \sin DCE \frac{CE}{DE} = \sin 27 \frac{(\cdot42)a}{(\cdot72)a} = \cdot45 . \tfrac{7}{12} = \cdot26;$$

therefore $\sin CDE = \sin 15°$. Now $EFC = DCB - CDE = 45° - 15° = 30°$.

15481. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize completely $N = 96^{18} + 1$.

Solution by the PROPOSER.

Let $$N = Y^{18} + 1 = \frac{Y^{18}+1}{Y^6+1} \cdot \frac{Y^6+1}{Y^2+1}(Y^2+1) = N_3 . N_1 . N_0 \text{ (suppose).}$$

Here taking $Y = 96$, gives $N_0 = 96^2 + 1 = 13 . 709$. Also, when $Y = 6y^2$,

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then N_1, N_3 are Sext-Aurifeuillians, and can be resolved into two factors L, M algebraically (by formulæ given in Ed. Lucas's *Sur la Série récurrente de Fermat*, Rome, 1879, p. 6) as follows:

$$N_1 = L_1 . M_1 ; \quad L_1 = P_1 - Q_1, \quad M_1 = P_1 + Q_1 ;$$
$$P_1 = 1 + 3Y_1 + 6Y_1^2, \quad Q_1 = 6y_1 (1 + Y_1).$$

Taking $y_1 = 4$ gives $Y_1 = 16$;

$$P_1 = 9505, \quad Q_1 = 3328 ; \quad L_1 = 7177, \quad M_1 = 11833 ;$$

therefore $N_1 = 7177 . 11833.$

Next, $Y^{18} = 96^{18} = 6^6 (6 . 4^3)^{12} = (6 . 384^2)^6 = Y_3^6 ;$

$$Y_3 = 6 . 384^2 ; \quad y_3 = 384.$$

Then $N_3 = (1 + Y_3^6)/(1 + Y_3^2)$, a Sext-Aurifeuillian.

$$N_3 = L_3 M_3, \quad L_3 = P_3 - Q_3, \quad M_3 = P_3 + Q_3 ;$$
$$P_3 = 1 + 3Y_3 + Y_3^2, \quad Q_3 = 6y_1 (1 + Y_3) ;$$

therefore $P_3 = 1 + 18 . 384^2 + 36 . 384^4 = 782,760,443,905 ;$

$$Q_3 = 6 . 384 (1 + 6 . 384^2) = 2,038,434,048 ;$$
$$L_3 = 700,722,009,857, \quad M_3 = 784,798,877,953 ;$$

Next a table of solutions of the congruence $(y^6 + 1)/(y^2 + 1) \equiv 0 \pmod{p}$ up to $p \not> 10,000$, prepared by the present writer (and which it is hoped will be published shortly), shows that N has the divisors 37, 73, 397, 613, and no more $< 10,000$. Here by actual trial division

$$L_3 = 37 . 397 . 53150113 ; \quad M_3 = 73 . 613 . 17537797 ;$$

and the large factors in L_3, M_3 must be primes (because the table alluded to shows that N_3 has no more divisors $\not> 10,000$).

Collecting the above results, $N = (L_3 M_3)(L_1 M_1) N_0$, and the factorization is complete. Note that N contains 36 figures.

15403. (A. M. Nesbitt, M.A.)—A triangle is inscribed in a circle, and the squares on its sides are proportional to the perpendiculars from the opposite angles on a tangent to the circle. Prove that one side is a diameter.

Solutions (I.) *by* R. F. Davis, M.A., Professor Joseph Brown, *and others;* (II.) *by* S. C. Bagchi, B.A., *and the* Proposer ; (III.) *by* Professor Sanjána, M.A.

(I.) Let AL, BM, CN be perpendiculars upon the tangent at any point P to the circum-circle of ABC. Then, since $PA^2 = 2R . AL$, and $BC . PA \pm CA . PB \pm AB . PC = 0$, therefore $\Sigma BC \surd(AL) = 0$, universally. In this case $\Sigma(\pm BC^2) = 0$; and the triangle ABC is right-angled.

(II.) Taking the inscribed triangle as that of reference, the equation to the circle is $a/x + b/y + c/z = 0$. The tangent at any point (f, g, h) of the circle is $lx + my + nz = 0$ where $lf^2 = a$, $mg^2 = b$, $nh^2 = c$.

Let $\Delta_1^2 = l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C$. By the conditions of the problem $\lambda a^2 = ap_1/(\Delta_1 f^2)$, $\lambda b^2 = bp_2/(\Delta_1 g^2)$, $\lambda c^2 = cp_3/(\Delta_1 h^2)$.

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But $ap_1 = bp_2 = cp_3 = 2(\text{area of } ABC)$; therefore $\pm af = \pm bg = \pm ch$. Now of f, g, h two must be of the same sign, Take h negative and f, g both positive. Then, since $a/f + b/g + c/h = 0$, $a^2 + b^2 = c^2$; therefore C is a right angle, and AB is a diameter.

(III.) If $\theta_1, \theta_2, \theta_3$ be the angles made by the lines joining angular points to the point of contact with the tangent, we readily get, if we suppose $\theta_3 > \theta_2 > \theta_1$,

$$\sin(\theta_3 - \theta_2)/\sin\theta_1 = \sin(\theta_3 - \theta_1)/\sin\theta_2 = \sin(\theta_2 - \theta_1)/\sin\theta_3.$$

Solving for $\sin^2\theta_3$ and $\sin\theta_3\cos\theta_3$, we obtain

$$\sin^2\theta_3 = \sin^2\theta_2 - \sin^2\theta_1, \quad \sin\theta_3\cos\theta_3 = \sin(\theta_2 - \theta_1)\cos(\theta_2 + \theta_1);$$

whence $\tan\theta_3 = \tan(\theta_2 + \theta_1)$, *i.e.* $\theta_3 - \theta_1 = \theta_2$. Also

$$\sin^2(\theta_2 + \theta_1) = \sin^2\theta_2 - \sin^2\theta_1$$

gives (as $\theta_2 + \theta_1 \neq 0$) $\sin(\theta_2 + \theta_1) - \sin(\theta_2 - \theta_1) = 0$, or $2\cos\theta_2\sin\theta_1 = 0$, which is satisfied by $\cos\theta_2 = 0$. Thus $\theta_3 - \theta_1 = \frac{1}{2}\pi$; so that the side joining the first and third points is a diameter.

15165. (Professor SANJÁNA, M.A.).—Show that, when $6n + 3$ is not a power of 3, $(x^{6n+3} + 1)/(x^3 + 1)$ has three factors at least, n being a positive integer and $3x$ a perfect square.

Note by the PROPOSER.

When $6n + 3 = 3^k$, I find, by ordinary methods, that the number of algebraical factors of $(x^{6n+3} + 1)/(x^3 + 1)$ is $2(k-1)$. Thus, for instance,

$$(x^9 + 1)/(x^3 + 1) = x^6 - x^3 + 1 = (x^3 - xy + 1)(x^3 + xy + 1),$$

where $y^2 = 3x$.

Also $(x^{27} + 1)/(x^3 + 1) = (x^6 - x^3 + 1)(x^{15} - x^9 + 1)$, and each bracket on the right side has two factors, as shown above. So also

$$(x^{81} + 1)/(x^3 + 1) = (x^6 - x^3 + 1)(x^{18} - x^9 + 1)(x^{54} - x^{27} + 1).$$

The process is clearly general.

15449. (JAMES BLAIKIE, M.A.)—If $u_1, u_2, u_3, \ldots, u_n$ be a series subject to the conditions (1) $u_1 = 1$, (2) $u_n = u_2 u_{n-1} + u_{n-2}$, prove the relations (3) $u_n = u_r u_{n-r+1} + u_{r-1} u_{n-r}$, (4) $u_{2r-1} = u_r^2 + u_{r-1}^2$,

(5) $u_{2r} = (u_{r+1}^2 - u_{r-1}^2)/u_2$, (6) $\Sigma_n u = (u_n + u_{n+1} - u_1)/u_2$.

This Question is in part a generalization of Question 15229. When $u_2 = 2$, the series becomes 1, 2, 5, 12, 29, ..., which is useful in connection with the solution of Question 15414.

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Solution by the PROPOSER.

By (2) $u_n = u_2u_{n-1} + u_{n-2} = u_2(u_2u_{n-2} + u_{n-3}) + u_{n-2}$

$= (u_2^2+1)u_{n-2} + u_2u_{n-3} = u_3u_{n-2} + u_2u_{n-3} = \ldots$

$= u_ru_{n-r+1} + u_{r-1}u_{n-r}$ ..... (3).

Substituting $2r-1$ for n, we obtain

$$u_{2r-1} = u_r^2 + u_{r-1}^2 \quad \ldots\ldots (4).$$

Substituting $2r$ for n,

$u_{2r} = u_ru_{r+1} + u_{r-1}u_r = (u_{r+1} + u_{r-1})(u_{r+1} - u_{r-1})/u_2$ by (2)

$= (u_{r+1}^2 - u_{r-1}^2)/u_2$ ..... (5).

Again by (2) we have

$u_{n+1} - u_{n-1} = u_2u_n$, $u_n - u_{n-2} = u_nu_{n-2}$, ..., $u_3 - u_1 = u_2u_2$, $u_2 = u_2u_1$.

Summing, we obtain

$$u_{n+1} + u_n - u_1 = u_2\Sigma_n u \text{ or } \Sigma_n u = (u_{n+1} + u_n - u_1)/u_2 \ldots\ldots (6).$$

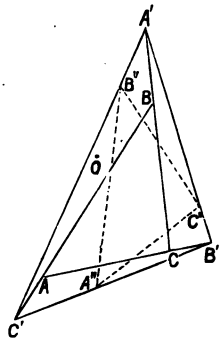
15273. (V. DANIEL, B.Sc.)—Required the conditions that two triangles may be mutually rotatable on to one another about a common point in their plane (A to lie on B′C′, A′ on BC, ...).

Solution by the PROPOSER.

Let ABC, A′B′C′ be two triangles, the angles of Δ' dividing the sides of Δ so that $BA' = ka$, $CB' = lb$, $AC' = mc$. Let it be possible for Δ to be rotated about O through an angle θ into the position A″B″C″, the angles of Δ'' dividing the sides of Δ' so that $B''A' = l'b'$, $C''B' = m'c'$, $A''C' = k'a'$. Then, the quantities k, l, m, k', l', m' being reckoned positive or negative according to external or nternal division of the sides, it is known that

$$\Delta'/\Delta = (1+k)(1+l)(1+m) - klm = \lambda^2,$$

$$\Delta''/\Delta' \equiv \Delta/\Delta' = (1+k')(1+l')(1+m') - k'l'm' = \lambda'^2.$$



The six quantities being connected by the relation $(\lambda^2\lambda'^2 = 1)$, it can be shown that

$$a^2/2\Delta = \{l(1+m') - l'(1+m)\} \operatorname{cosec}\theta, \text{ and two others} \ldots\ldots (1),$$

$$a'^2/2\Delta' = \{m(1+l') - m'(1+l)\} \operatorname{cosec}\theta, \text{ and two others} \ldots\ldots (2).$$

Hence $\cot\omega + \cot\omega' = (\Sigma k - \Sigma k')\operatorname{cosec}\theta$,

$\cot\omega - \cot\omega' = (\Sigma lm' - \Sigma l'm)\operatorname{cosec}\theta$;

therefore $2(\cot^2\omega + \cot^2\omega') = \{(\Sigma k - \Sigma k')^2 + (\Sigma lm' - \Sigma l'm)^2\}\operatorname{cosec}^2\theta$;

therefore

$$2(\Sigma \operatorname{cosec}^2 A + \Sigma \operatorname{cosec}^2 A' - 2) = \{(\Sigma k - \Sigma k')^2 + (\Sigma lm' - \Sigma l'm)^2\}\operatorname{cosec}^2\theta;$$

i.e., $2(\Sigma a^2b^2/4\Delta^2 + \Sigma a'^2b'^2/4\Delta'^2 - 2) = \{(\Sigma k - \Sigma k')^2 + (\Sigma lm' - \Sigma l'm)^2\}\operatorname{cosec}^2\theta$;

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therefore $4\sin^2\theta = \big[2\Sigma\{l(1+m')-l'(1+m)\}\{n(1+k')-m'(1+k)\}$
$+2\Sigma\{m(1+l')-m'(1+l)\}\{k(1+m')-k'(1+m)\}$
$-\{\Sigma k-\Sigma k'\}^2-\{\Sigma lm'-\Sigma l'm\}^2\big]$.

To prove the fundamental relations (1) and (2), we have, from the triangles A′B′C and A″B″C′,

$$c'^2 = (1+k)(1+k+l)\,a^2 + l(1+k+l)\,b^2 - l(1+k)\,c^2 \quad\ldots\ldots\ldots\ldots\ldots(3),$$
$$c^2 = k'(1+k'+l')\,a'^2 + (1+l')(1+k'+l')\,b'^2 - k'(1+l')\,c'^2 \quad\ldots\ldots(4).$$

From (3) and its two cyclic changes, we get, solving for a^2, b^2, c^2,

$$c^2\lambda^4 = k(1+k+l)\,a'^2 + (1+l)(1+k+l)\,b'^2 - k(1+l)\,c'^2 \quad\ldots\ldots\ldots(5).$$

Similarly,

$$c'^2\lambda'^4 = (1+k')(1+k'+l')\,a^2 + l'(1+k'+l')\,b^2 - l'(1+k')\,c^2 \quad\ldots\ldots(6),$$

because (3) and (4) are transformable with

$$[1+k+k' = 1+l+l' = 1+m+m' = 0].$$

Combining (3) and (6) and also (4) and (5) with the help of $\lambda^2\lambda'^2 = 1$, we get

$$\{(1+k)(1+k+l)\lambda'^2 - (1+k')(1+k'+l')\lambda^2\}\,a^2$$
$$+\{l(1+k+l)\lambda'^2 - l'(1+k'+l')\lambda^2\}\,b^2$$
$$-\{l(1+k)\lambda'^2 - l'(1+k')\lambda^2\}\,c^2 = 0 \quad\ldots\ldots\ldots\ldots(7),$$

$$\{k'(1+k'+l')\lambda^2 - k(1+k+l)\lambda'^2\}\,a'^2$$
$$+\{(1+l')(1+k'+l')\lambda^2 - (1+l)(1+k+l)\lambda'^2\}\,b'^2$$
$$-\{(1+l')k'\lambda^2 - (1+l)k\lambda'^2\}\,c'^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots(8).$$

These, being solved simultaneously with their cyclic changes and simplified by the use of the identities

$$(1+m)(1+k+l)+kl = \ldots = \ldots = \lambda^2 = m(1+k+l)+(1+k)(1+l) = \ldots = \ldots,$$

give finally, after cancelling a large factor,

$$a^2 : b^2 : c^2 = \{l(1+m')-l'(1+m)\} : \{n(1+k')-m'(1+k)\}$$
$$: \{k(1+l')-k'(1+l)\} \quad\ldots\ldots(9),$$
$$a'^2 : b'^2 : c'^2 = \{m(1+l')-m'(1+l)\} : \{k(1+m')-k'(1+m)\}$$
$$: \{l(1+k')-l'(1+k)\}\ldots\ldots(10).$$

Substituting in (3) the values $a^2 = \{l(1+m')-l'(1+m)\}\,x\Delta$, ..., gives $c'^2/x\Delta = \{l(1+k')-l'(1+k)\}\,\lambda^2$; therefore

$$a'^2 = \{n(1+l')-m'(1+l)\}\,x\Delta', \ \ldots\ .$$

θ being the angle between corresponding sides of Δ and Δ'', can be written in three ways as the sum of two angles, *e.g.*, $\theta = \mathrm{B'A'C} + \mathrm{B''C''A'}$.

But
$$\sin \mathrm{B'A'C} = lb\sin \mathrm{C}/c' = 2l\Delta/ac',$$
$$\sin \mathrm{B''C''A'} = -l'b'\sin \mathrm{A}'/a = -2l'\Delta'/ac'.$$

Again, $2(1+k)\,ac'\cos \mathrm{B'A'C} = (1+k)^2a^2 + c'^2 - l^2b^2$
$= 2(1+k)^2a^2 + l(1+k)(a^2+b^2-c^2)$, by (3),
and $2(1+m')\,c'a\cos \mathrm{B''C''A'} = (1+m')^2c'^2 + a^2 - l'^2b'^2$
$= 2(1+m')^2c'^2 + l'(1+m')(b'^2+c'^2-a'^2)$, by (4)
therefore $a^2c'^2\sin \mathrm{B'A'C}\cos \mathrm{B''C''A'} = l\Delta\{2(1+m')\,c'^2 + l'(b'^2+c'^2-a'^2)\}$,
$a^2c'^2\cos \mathrm{B'A'C}\sin \mathrm{B''C''A'} = -l'\Delta'\{2(1+k)\,a^2 + l(a^2+b^2-c^2)\}$;

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therefore $a^2c'^2 \sin\theta/x\Delta\Delta' = l/x\Delta' \{2(1+m')c'^2 + l''(b'^2+c'^2-a'^2)\}$
$-l'/x\Delta \{2(1+k)a^2 + l(a^2+b^2-c^2)\}$.

Substituting from (9) and (10) and reducing, we get
$a^2c'^2 \sin\theta/2x\Delta\Delta' = \{l(1+m') - l'(1+m)\}\{l(1+k') - l'(1+k)\} = a^2c'^2/x^2\Delta\Delta'$;
therefore $x \sin\theta = 2$. Hence equations (1) and (2) and the expression for $\sin^2\theta$.

15493. (*Communicated by* C. M. Ross.)—Two straight lines move between two rectangular axes in such a way that their ends are always concyclic. Find the locus of the centre of the circle through the ends of these lines. (Solution by either pure or analytical geometry.)

Solutions (I.) *by* Professor Jan de Vries *and the* Proposer ; (II.) *by* Frances E. Cave.

(I.) If the lines are denoted by a and b, the segments which they determine on the rectangular axes OX and OY may be denoted by $a\cos\theta$, $a\sin\theta$ and by $b\sin\theta$, $b\cos\theta$. Hence the centre of the circle is determined by $2X = a\cos\theta + b\sin\theta$, $2Y = a\sin\theta + b\cos\theta$. Eliminating θ, we find that the locus of the centre of the circle is the ellipse with the equation $(a^2+b^2)(x^2+y^2) - 4abxy = \frac{1}{2}(a^2-b^2)^2$.

(II.) Let O be intersection of axes; AB, CD the two lines. Complete the rectangles OBEA, ODFC: then centre P of circle is middle point of FE. Also, since $\angle BAC = \angle BDC$, OF and OE make equal angles θ with bisector of $\angle AOB$. Therefore, if a and b be given lengths, co-ordinates of P referred to bisectors of $\angle AOB$ are $\frac{1}{2}(a+b)\cos\theta$ and $\frac{1}{2}(a-b)\sin\theta$, and therefore equation of locus is

$$x^2/[\tfrac{1}{2}(a+b)]^2 + y^2/[\tfrac{1}{2}(a-b)]^2 = 1.$$

15214. (D. Biddle.)—Having given that $x^2-y^2 = a$ and $xz-yw = b$, find z^2-w^2, all the values being integral, but (except a and b) unknown.

Solution by Lt.-Col. Allan Cunningham, R.E.

Let a be resolved into its factors; these will $= (x \mp y)$. Every resolution of a into a pair of factors gives one set of values of x, y. Hence z, w may be found in the equation $xz-yw = b$ (when possible); and hence also (z^2-w^2).

[The Proposer remarks that, corresponding to b is b', that is, $xw-yz$. Moreover, $z^2-w^2 = (b^2-b'^2)/a$. Therefore $b'^2 \equiv b^2 \pmod{a}$. We also have $b+b' = (x-y)(z+w)$, whilst $b-b' = (x+y)(z-w)$. Consequently, all the unknown quantities, including b', can be found with comparative ease.]

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15440. (H. Bateman, B.A.)—A sextic curve of the third class has cusps at the circular points and seven other cusps: these may be divided into sets of type A, B, C, such that the tangents at A, B, C meet in a point O. Prove that the Euler line of the triangle ABC passes through O and also through the triple focus.

Solution by R. F. Whitehead, B.A.

Take the sextic in the canonical form (using line co-ordinates ξ, η, ζ) $\xi^3+\eta^3+\zeta^3+6l\xi\eta\zeta=0$. Let the circular points at infinity be I $(1, \omega, -2l\omega^2)$ and J $(1, \omega^2, -2l\omega)$, so that the triple focus S is (0, 0, 1). The third cusp, the tangent at which passes through S, is the point C $(1, 1, -2l)$. On the line CS three other points O, P, Q lie, through each of which the tangents from two of the nine cusps pass. These points are O (1, 1, 1), P $(1, 1, \omega)$, and Q $(1, 1, \omega^2)$. The cusps, the tangents at which pass through O, are C $(1, 1, -2l)$, A $(-2l, 1, 1)$, and B $(1, -2l, 1)$.

The line BC is $x(2l-1)+y+z\equiv X=0$, CA is $x+y(2l-1)+z\equiv Y=0$, AB is $x+y+z(2l-1)\equiv Z=0$. The polar circle for the triangle ABC is $(X^2+Y^2)(1-2l+4l^2)^2+12Z^2(1-2l-2l^2)^2\equiv S=0$. The line infinity, *i.e.*, the line IJ, is $4l(X+Y)-(2l^2-2l+1)Z=0$. The circum-circle is (for special values of p, q, r) $pYZ+qZX+rXY\equiv T=0$. For a certain value of θ, $S+\theta T=0$ represents the line infinity, and the radical axis of the nine-point and circum-circles. Hence, if $S\equiv AX^2+BY^2+CZ^2=0$, and the line infinity $\equiv LX+MY+NZ=0$, the point at infinity on the radical axis is given by the meet of

$$LX+MY+NZ=0 \quad \text{and} \quad AX/L+BY/M+CZ/N=0,$$

and its pole with respect to the self-polar circle, which is the line of centres, *i.e.*, the Euler line (since the centre of the polar circle is the orthocentre), is

$$AL(BN^2-CM^2)X+BM(CL^2-AN^2)Y+CN(AM^2-BL^2)Z=0,$$

or $X-Y=0$, *i.e.*, $x-y=0$, which is the line CS. Now O may be taken as any one of the three meets of cuspidal tangents on the line CS. Hence the theorem follows that there are three sets of cusps A, B, C whose tangents meet in a point O, whose join to the triple focus S is the Euler line of the triangle ABC.

Note by the Proposer.—The curve appears to be symmetrical about the line CS: this can easily be proved to be the case.

9825. If particles be projected along lines meeting in a point, with velocities proportional to the projections of the same vertical line upon each line, at any time the particles will lie upon the sphere described upon the space traversed by the vertical particle as diameter. Hence

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find the line of quickest transit when an inelastic particle is dropped from O to A (a vertical distance h), and after impact upon an inelastic line AB (the line to be determined) proceeds along it to B, a point on a given circle in a vertical plane through OA.

Solution by A. M. Nesbitt, M.A.

If the particles are projected upwards, V being the velocity of the vertical one, the space traversed in time t by one projected at an angle θ with the vertical will be given by $OP = Vt\cos\theta - \frac{1}{2}g\cos\theta . t^2$. But

$$OA = Vt - \tfrac{1}{2}gt^2;$$

therefore $OP = OA\cos\theta$ or P lies on the sphere which has OA for diameter. If they are projected downwards, the $-$ signs are changed to $+$, but otherwise the reasoning applies. Also when the particle has reached A it will proceed along AB with velocity $V\cos\theta$, where $V^2 = 2gh$ and θ is the inclination of AB to the vertical. Hence to get the line of quickest descent along OAB we must describe a circle with its centre in OA produced passing through A, and touching the given circle in B.

15464. (C. M. Ross.)—Prove that the continued fraction

$$\frac{a}{(a+1)-}\ \frac{b}{(b+1)-}\ \frac{c}{(c+1)-}\ \ldots \text{ ad inf.}$$

has the value unity if a, b, c, ... are positive integers.

Solutions (I.) *by* S. C. Bagchi, B.A.; (II.) *by the* Proposer.

(I.) Let the n-th fraction be $n/(n+1)$, the $(n-1)$-th be $m/(m+1)$, ... $p_n = (n+1)p_{n-1} - np_{n-2}$; therefore

$$p_n - p_{n-1} = n(p_{n-1} - p_{n-2}) = nm(p_{n-2} - p_{n-3}) = \ldots = abcd\ldots mn$$

or $p_n = abc\ldots mn + abc\ldots m + \ldots + p_1$. Similarly $q_n = abc\ldots m + \ldots + q_1$. But $p_1 = a$, $q_1 = a+1$; therefore $p_n/q_n = S_n/(S_n+1)$, where

$$S_n = a + ab + abc + \ldots + abc\ldots mn.$$

The sum to infinity $= \underset{n=\infty}{\mathrm{L}}\, p_n/q_n = \underset{n=\infty}{\mathrm{L}}\, 1/(1+1/S_n)$. But, since a, b, c, ... are positive integers, the limit of the series $a + ab + abc + \ldots$ is clearly greater than any given quantity or $\underset{n=\infty}{\mathrm{L}}\, S_n = \infty$; therefore the continued fraction tends to the limit unity.

(II.) The first three convergents are

$$\frac{a}{a+1},\quad \frac{a(b+1)}{a(b+1)+1},\quad \frac{a(bc+b+1)}{a(bc+b+1)+1}.$$

Hence $q_1 = 1 + p_1$, $q_2 = 1 + p_2$, $q_3 = 1 + p_3$. Now suppose the relation holds for the $(n-1)$-th and n-th convergents, *i.e.*, suppose $q_{n-1} = 1 + p_{n-1}$ and $q_n = 1 + p_n$. By the recurrence formulæ

$p_{n+1} = (l+1)p_n - lp_{n-1}$, where $l/(l+1)$ is the $(n+1)$th fraction,

$q_{n+1} = (l+1)q_n - lq_{n-1} = (l+1)(1+p_n) - l(1+p_{n-1})$ by hypothesis

$= (l+1)p_n - lp_{n-1} + 1 = p_{n-1} + 1.$

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Hence the relation holds for the numerator and denominator of the $(n+1)$-th convergent if it is true for the n-th and $(n-1)$-th convergents. But it does hold for the 2nd and 3rd; therefore it is true universally.

Again the value of the fraction

$$= \underset{n=\infty}{\mathrm{L}}\, p_n/q_n = \underset{n=\infty}{\mathrm{L}}\, (q_n-1)/q_n = 1.$$

15451. (A. Ensor.)—Find the form of n if the greatest coefficient in the expansion of $(1+x)^n$ is a multiple of $n(n-2)$.

Solution by Lt.-Col. Allan Cunningham, R.E.

If n, $n-2$ be *primes* (>5), then the $(r+1)$-th coefficient is $n!+r!\,(n-r)!$, which is a multiple of $n(n-2)$ for all values of $r>2$ and $<n-1$. This includes the greatest coefficient.

15460. (S. C. Bagchi, B.A.)—A triangle moves so that its base is always parallel to the axis of a semi-cubical parabola, while its two sides and the median passing through their intersection are three tangents to that cubic. Prove that the locus of the vertex is the orthogonal plane projection of the same semi-cubical parabola, the axis being the axis of projection.

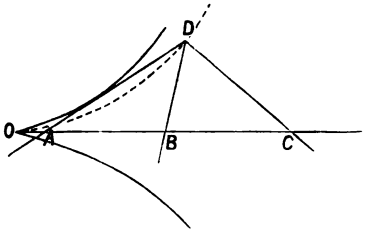
Solution by the Proposer.

Let the cusp of the semi-cubical parabola be at $x=2a$, from the origin. Since each of the straight lines DA, DB, DC touches the cubic, the equation to any of these may be written

$$y = mx - 2am - am^3,$$

m being different for each of the lines. Because

$$AB = BC,$$



therefore $2m_2^2 = m_1^2 + m_3^2$ .....(i.),

$$\Sigma m_1 = 0, \quad a\Sigma m_1 m_2 = x-2a, \quad m_1 m_2 m_3 a = y \text{ ...(ii., iii., iv.).}$$

Eliminating m_1, m_2, m_3 from these equations, we get for the locus in question $27ay^2 = 2(x-2a)^3$. And the equation to the given cubic is $27ay^2 = 4(x-2a)^3$; therefore the locus is also a semi-cubical parabola whose ordinates are diminished in the ratio $1 : \sqrt{2}$. The dotted line in the figure represents the curve.

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Mr. R. F. Davis, M.A., remarks that the theorem may also be stated as follows:—If three concurrent normals intersect the axis of a parabola in G_1, G_2, G_3 and $G_1G_2 = G_2G_3$, then the locus of the point of concurrence is the curve formed by diminishing the ordinates of the evolute in the ratio $1 : \sqrt{2}$.

15461. (Professor Neuberg.)—Couper un trièdre donné SABC par un plan qui rencontre les arêtes en des points A, B, C, tels que les projections de A sur SB et SC, de B sur SC et SA, de C sur SA et SB soient six points d'une même sphère.

Solution by A. M. Nesbitt, M.A.

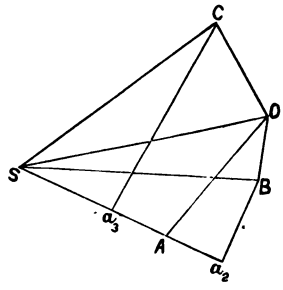
Take SA, SB, SC for oblique axes of co-ordinates and let the angles BSC, CSA, ASB be denoted by α, β, γ. Then, if Ab_1, Ac_1 be the perpendiculars from A on SB, SC; Bc_2, Ba_2 from B on SC, SA; and Ca_3, Cb_3 from C on SA, SB, we have

$$Sa_2 = SB\cos\gamma, \quad Sa_3 = SC\cos\beta;$$

so that, if $SA = x$, $SB = y$, $SC = z$,

$$yz\cos\beta\cos\gamma = zx\cos\gamma\cos\alpha = xy\cos\alpha\cos\beta$$

or $x\cos\alpha = y\cos\beta = z\cos\gamma$. Hence, if O be any point on the line given by these equations, and x, y, z be its co-ordinates, the equation to the required plane will be $\xi/x + \eta/y + \zeta/z = 1$; which, if we are given one of the three points, say A, becomes



$$(\xi - x)\cos\alpha + \eta\cos\beta + \zeta\cos\gamma = 0.$$

15482. (James Blaikie, M.A.)—Find two numbers of eleven digits such that in each case the square of the number is terminated by the eleven digits of the number; and show that the sum of these two numbers is necessarily $10^{11} + 1$.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.;
(II.) *by* Rev. T. Wiggins, B.A.

(I.) It can be seen by inspection of a table of squares that the only numbers (N) of n digits ($n > 1$) possessing the property required, viz., that N^2 ends in the n digits composing N, or, in other words, that $N^2 - N = 10^n$. M, are numbers ending in 25 and 76. Next, it can be

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seen by inspecting a table of powers of 5 that (5^4 and 5^8), (5^8 and 5^{16}), (5^{16} and 5^{32}), ..., and finally (5^m and 5^{2m}, where $m = 2^\mu$) have the 4, 5, 6, ..., $(\mu+2)$ right-hand digits common. Hence the number N_n, consisting of the n right-hand digits of $5^{2^{n-2}}$, is a number whose square ends in those same digits, and is therefore one of the numbers sought. It is not necessary to form the complete powers of 5 up to the limit 2^{n-2} stated; it suffices to square the number N_x of x figures yielded by 5^{2x-2} to obtain a number N_{x+1} correct to $x+1$ figures, and then squaring yields N_{x+2}, and so on.

E.g., $5^{2^3} = 5^8 = 390625$, this gives $N_5 = 90625$; $N_5^2 = 8212890625$, this gives $N_6 = 890625$. Proceeding thus, finally, $N_{11} = 18212890625$ (one of the required numbers). Next, let $N'_n = 10^n + 1 - N_n$. Then

$$(N'_n)^2 = 10^{2n} + 2 \cdot 10^n + 1 + N_n^2 - 2 \cdot 10^n \cdot N_n - 2N_n$$

$$= 10^n(10^n + 1 - 2N_n) + (N_n^2 - N_n) + (10^n + 1 - N_n).$$

But $N_n^2 - N_n \equiv 0 \pmod{10^n}$ by definition, and $10^n + 1 - N_n = N'_n$; therefore $(N'_n)^2 - N'_n \equiv 0 \pmod{10^n}$; so that N'_n is also one of the numbers required; hence $N'_{11} = 81787109376$.

Note.—The theory of these numbers has been developed in a paper on "Agreeable Numbers"; an abstract is given in *Brit. Assoc. Report*, 1893, p. 699. Note that $N_n^r - N_n \equiv \pmod{10^n}$ for all values of r. Also that $(N_n - 1)^r - (N_n - 1) \equiv 0 \pmod{10^n}$ for all odd values of r; see the paper quoted.

(II.) Let $x^2 = 10^{11}y + x$; therefore $10^{11}y = x(x-1)$. Whence it is easily seen:—(i.) that neither factor is a multiple of 10; (ii.) that one factor is odd and a multiple of 5; (iii.) that one factor is even and not a multiple of 5. Thus the factors are of the form $5^{11}(2q+1)$ and $2^{11}p$; therefore $x_1 = 5^{11}(2q_1+1) = 2^{11}p_1 + 1$, $x_2 = 5^{11}(2q_2+1) + 1 = 2^{11}p_2$; therefore $S = x_1 + x_2 = 5^{11}(2q_1 + 2q_2 + 2) + 1 = 2^{11}(p_1 + p_2) + 1$;

therefore $\frac{1}{2}(S-1) = 5^{11}(q_1 + q_2 + 1) = 2^{10}(p_1 + p_2)$.

Now 5^{11} is an 8-figured number (48828125); therefore $(q_1 + q_2 + 1)$ must be less than 3000 and a multiple of 2^{10} (2048), because $\frac{1}{2}(S-1)$ has not more than 11 figures; therefore $q_1 + q_2 + 1 = 2^{10}$; therefore

$$S - 1 = 2 \cdot 2^{10} \cdot 5^{11} \quad \text{or} \quad S = 10^{11} + 1.$$

To find the numbers solve $48828125(2q+1) = 2048p + 1$, which gives $p = 8893013$; therefore $x_1 = 18212890625$; hence $x_2 = 81787109376$.

Note by the Proposer on Solution (I.).—It can easily be proved, without reference to tables, that, where $m = 2^\mu$, 5^{2m} and 5^m have the $\mu+2$ right-hand digits common. We have

$$5^{2m} - 5^m = 5^m(5^m - 1) = 5^m(5-1)(5+1)(5^2+1)\ldots.$$

The first factor in brackets is 2^2, and there are μ other factors each of which is oddly even. Therefore $5^{2m} - 5^m$ is a multiple of $10^{\mu+2}$.

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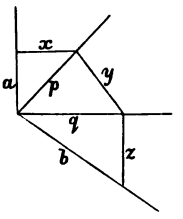
Propositions connected with Spirals.

By THEODORE W. HAY.

Theorem I.—If the angle between a and b be made up of three equal angles between the lines a, p, q, and b, and from any point in a a straight line x perpendicular to a be drawn to meet p, and from the point of intersection of p and x a straight line y perpendicular to p be drawn to meet q, and from the point of intersection of q and y a straight line z perpendicular to q be drawn to meet b, then



$$a^2/b^2 = x^3/z^3.$$

$$b^2 = q^2 + z^2 = z^2 + y^2 + p^2 = z^2 + y^2 + x^2 + a^2;$$

therefore $b^2 - a^2 = z^2 + y^2 + x^2$ .....(i.).

The triangles axp, pyq, qzb are similar; therefore $x/p = y/q$, *i.e.*, $x/y = p/q$ and also $y/z = p/q$; therefore $x/y = y/z$, and $y^2 = xz$.

Hence $b^2 - a^2 = z^2 + xz + x^2$ .....(ii.).

Now $x^2/a^2 = z^2/q^2 = z^2/(b^2 - z^2)$; therefore

$$a^2 = [x^2(b^2 - z^2)]/z^2 \quad \text{and} \quad b^2 - [x^2(b^2 - z^2)]/z^2 = x^2 + xz + z^2.$$

From this we get at once $b^2(z^2 - x^2) = z^3(x + z)$ or

$$b^2 = z^3/(z - x) \text{ .....(iii.).}$$

Also, from (ii.), $a^2 = b^2 - z^2 - xz - x^2$, *i.e.*,

$$a^2 = z^3/(z - x) - z^2 - xz - x^2 = x^3/(z - x) \text{ .....(iv.).}$$

Therefore, combining (iii.) and (iv.), $a^2/b^2 = x^3/z^3$.

Note.—The formula $a^2/b^2 = x^3/z^3$ holds good whatever angle a and x make; for $q = \sqrt{(ab^2)}$ (being the second of two means between a and b); now $a/x = q/z = \sqrt[3]{(ab^2)}/z$; therefore $a/\sqrt{(ab^2)} = x/z$; therefore

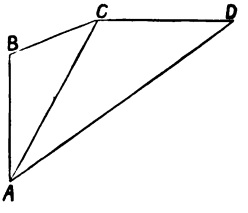
$$a^3/ab^2 = x^3/z^3 = a^2/b^2,$$

without reference to the angle between a and x.

Theorem II.—The formula $a^2/b^2 = x^3/z^3$ affords a proof of the following theorem: that the sum and difference of the parts of a line divided in extreme and mean ratio are as the cubes of the parts.

Lemma.—In the triangle ABC let

$$AC \cdot BC = AB^2.$$



If from C a straight line CD be drawn making $\angle ACD = \angle B$, and CD be taken equal to AB, then the triangle formed by joining AD will be similar to the triangle ABC and $\angle CAD = \angle BAC$.

By hypothesis $AC \cdot CB = AB^2 = CD^2$; therefore $AC/CD = CD/CB = AB/CB$, also $\angle ACD = \angle B$; therefore the triangles ABC, ACD are similar, and, because $AC/CD = AB/BC$, therefore

$$\angle CAD = \angle BAC.$$

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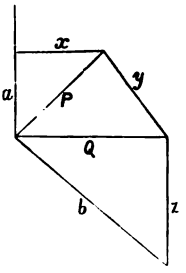
In the diagram let the angles between a and x, p and y, q and z be right angles, and let $a = y$;



then $b^2 = z^2 + y^2 + x^2 + a^2 = z^2 + 2y^2 + x^2$

$= z^2 + 2xz + x^2$;

therefore $b = x + z$ .....(1).

Again, $a^2 = y^2 = xz$; but $a^2 = x^2/(z-x)$ as shown above; therefore $xz = x^2/(z-x)$ or $z = x^2/(z-x)$. This evidently gives $xz = (z+x)(z-x)$. But, by (1), $b = x + z$; therefore $zx = b(z-x)$, or

$$z - x = xz/b = a^2/b \text{.....(2).}$$

Now, since $z^2 - x^2 = xz$, therefore $(z+x)x = z^2$, *i.e.*, $(z+x)$ is a line cut in extreme and mean section, x and z being the parts, and $z > x$.

Combining (1) and (2), $(z+x)/(z-x) = b(a^2/b) = b^2/a^2 = z^3/x^3$.

Verification of the above theorem by the actual ratio of the parts :— If $x + z$ be taken to be equal to unity, then $x = \frac{1}{2}(3 - \sqrt{5})$, $z = \frac{1}{2}(\sqrt{5} - 1)$, x and z being as the parts of a line divided in extreme and mean section.

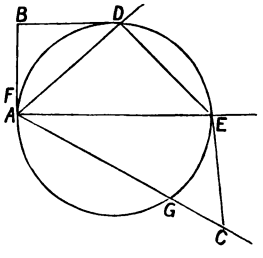
$$\frac{z+x}{z-x} = \frac{(\sqrt{5}-1)+(3-\sqrt{5})}{(\sqrt{5}-1)-(3-\sqrt{5})} = \frac{1}{\sqrt{5}-2}.$$

Also $\left(\frac{\sqrt{5}-1}{3-\sqrt{5}}\right)^3 = \frac{\sqrt{5}-2}{9-4\sqrt{5}}$. But $\frac{\sqrt{5}-2}{9-4\sqrt{5}} \; \frac{\sqrt{5}-2}{1} = 1$. Therefore

$\frac{1}{2}(\sqrt{5}-1) + \frac{1}{2}(3-\sqrt{5}) : \frac{1}{2}(\sqrt{5}-1) - \frac{1}{2}(3-\sqrt{5}) = (\sqrt{5}-1)^3 : (3-\sqrt{5})^3$.

Theorem III.—The angle BAC is made up of three equal angles BAD, DAE, EAC. $\angle B = \angle ADE = \angle AEC$, the angle B being any angle. AB, AD, AE, AC are in geometrical progression. AFG is a circle described about the triangle ADE, cutting AB in F and AC in G. To be shown that

$BF = \sqrt[3]{(AB)}$ and $CG = \sqrt[3]{(AC)}$.



N.B.—By the expression $BF = \sqrt[3]{(AB)}$ is meant that $BF : AB = \sqrt[3]{x} : x$, where x is some number or algebraic quantity.

Since the triangles are similar, and $\angle ADB = \angle AED$, and $\angle C = \angle AED$, the circle AFG is touched by BD in D and by EC in E. Let $BF = AB^n$ and $CG = AC^m$. Then $AB \cdot AB^n = BD^2$, and $AC \cdot AC^m = CE^2$; or $BD^2 = AB^{n+1}$ and $CE^2 = AC^{m+1}$; therefore $AB^{n+1}/AC^{m+1} = BD^2/CE^2$; also it has been shown that $AB^2/AC^2 = BD^3/CE^3$. Therefore

$$\frac{AB^{n+1}}{AC^{m+1}} \; \frac{AB^{\frac{3}{2}}}{AC^{\frac{3}{2}}} = \frac{BD^2}{CE^2} \; \frac{BD}{CE} = \frac{BD^3}{CE^3}, \quad \textit{i.e.}, \quad \frac{AB^{n+1}}{AC^{m+1\frac{1}{2}}} = \frac{AB^2}{AC^2};$$

therefore $n = m = \frac{1}{2}$; therefore $BF = \sqrt[3]{(AB)}$ and $CG = \sqrt[3]{(AC)}$, if $BF : AB = \sqrt[3]{x} : x$ and $CG : AC = \sqrt[3]{y} : y$.

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15379. (Rev. Prebendary WHITWORTH, M.A.) — In how many different orders can the eight digits 3, 3, 3, 3, 7, 7, 7, 7 be arranged so as to express a number divisible by 37 ?

Solution by SOLIDUS.

As there are but 70 different numbers formed from the digits 7, 7, 7, 7, 3, 3, 3, 3 it would not take long to test each separately as to divisibility by 37. But the following analysis shortens the actual arithmetical work and shows that there are 18 such numbers divisible by 37, and no more. Since $7003 - 3007 = 3996 = 37 \times 108$, it follows that, if in any number we interchange a 3 and a 7 which are separated by two intervening digits, we do not alter the remainder of the number when divided by 37. Hence in a number consisting entirely of 7's and 3's the digits in the 1st, 4th, and 7th places may be permuted amongst one another without altering the divisibility by 37. Let us call these three places "the place-group I." Similarly the place-group II will consist of the 2nd, 5th, and 8th places, and place-group III of the 3rd and 6th places. All the possible numbers formed by 7, 7, 7, 7, 3, 3, 3, 3 might be classified as follows, according to the number of 7's in each place-group :—

I.....	3	3	2	2	2	1	1	1	0	0
II	1	0	2	1	0	3	2	1	3	2
III.....	0	1	0	1	2	0	1	2	1	2

Of these nine types of numbers it is easy by taking a specimen of each to verify that the only type divisible by 37 is that in which place-group I contains one 7, place-group II two 7's, and place-group III one 7, and that there are 18 permutations possible within the group. These are the numbers :—

33337777, 33377737, 73337737, 33737377, 33777337, 73737337,
37337773, 37373737, 77333737, 37733377, 37773337, 77733337,
37333777, 37377733, 77337733, 37737373, 37777333, 77737333.

15173. (Professor NEUBERG.)—Si $\tan\alpha, \tan\beta, \tan\gamma$ sont les racines de l'équation $ax^3 + 3bx^2 + 3cx + d = 0$, former l'équation qui a pour racines $\tan\alpha \tan(\beta+\gamma)$, $\tan\beta \tan(\gamma+\alpha)$, $\tan\gamma \tan(\alpha+\beta)$.

Solution by R. TUCKER, M.A., F. W. REEVES, B.A., *and* M. RAMPAL.

Proceeding in the usual way, we get

$y = \tan\alpha \tan(\beta+\gamma) = x[(-3b/a) - x]/(1 + d/ax) = -(ax^3 + 3bx^2)/(ax + d)$,

i.e., $y = (3cx + d)/(ax + d)$: hence x is linear in y, and the substitution $x = d(y-1)/(3c - ay)$ gives the required equation in y.

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15008. (D. Biddle.)—A hollow sphere of radius unity has concentric to it a smaller sphere of radius ρ. Find the greatest number of spheres that can be made to occupy the intervening space, each having internal contact with the one and external contact with the other original sphere.

Solution by J. P. Hjorth.

Imagining cones put with their tops at the centre of the original spheres, such that each of them is touching one of the added spheres, the number sought is equal to the number of circles of radius v which can be drawn on the surface of the sphere without cutting one another, where $\sin v = (1-\rho)/(1+\rho)$. Imagining the surface entirely covered with such circles, viz., when each is touching four others, and drawing the spherical tangents at the points of contact, the surface is divided into squares whose sides are equal to $2v$ and whose angles are $2x$, where

$$\tan x = \tan v/\sin v = (1+\rho)/(2\sqrt{\rho});$$

the area of such a square is $t = \frac{1}{720}(8x-360)4\pi$, and for the determination of n, the required number, we have $nt \leqslant 4\pi$, *i.e.*,

$$n(8x-360) \leqslant 720, \quad n \leqslant 720/(8x-360);$$

therefore the required number is the greatest integer in $720°/(8x-360°)$.

[The Proposer observes that the above solution seems correct, except in making each added sphere be in contact with a restricted number of other similar ones, namely, four. This would be the case for a particular value of ρ only; as ρ approached unity, and the added spheres became infinitesimal, their centres would be arranged hexagonally.]

14735. (Professor Morley.)—Given five planes $a_1, \ldots, a_5$, and a quadric Q, let a quadric Q_n pass through the intersection of Q and a_n ($n = 1, \ldots, 5$) and the four intersections of the four other planes. Then Q_n intersects Q in a second plane β_n, whose polar as to Q is a point a_n. Prove that the line a_1a_2 passes through the point (a_3, a_4, a_5); or, reciprocally, that the line a_1a_2 lies in the plane $a_3a_4a_5$. Also prove that, if one of the points a lies on Q, then all the points a lie on Q. Arrange the proof so as to hold for the analogous theorem in n dimensions. [The theorem in two dimensions that, given four lines and a conic C, and conics C_n being passed through the three intersections of three lines and the intersections of C and the fourth line, then, if one C_n touches C, all C_n's touch C, is M. Humbert's, *Comptes Rendus*, Vol. cxxix.; and an elegant elementary proof is given by M. Bricard, *Nouv. Ann.*, Sér. 3, Vol. xix., p. 369.]

Solution by A. M. Nesbitt, M.A.

Take $Q \equiv (abcdfghpqr)(xyz\omega)^2 = 0$. Let the five planes $a_1, \ldots, a_5$ be $x = 0$, $y = 0$, $z = 0$, $\omega = 0$, and $Ax + By + Cz + D\omega = 0$; then

$$Q_5 \equiv Q - a_5L, \quad \text{where} \quad L \equiv \alpha x + \beta y + \gamma z + \delta\omega;$$

$$Q_4 \equiv Q - \omega L', \quad \text{where} \quad L' \equiv \alpha' x + \ldots + \delta'\omega; \ \ldots.$$

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Now Q_5 passes through $yz\omega$, $zx\omega$, $xy\omega$, and xyz; therefore $\alpha = a/A$, $\beta = b/B$, .. ; therefore the condition that L should touch Q is

$$\begin{vmatrix} a & h & g & p & a/A \\ h & b & f & q & b/B \\ g & f & c & r & c/C \\ p & q & r & d & d/D \\ a/A & b/B & c/C & d/D & . \end{vmatrix} = 0,$$

which is therefore the condition that a_5, the pole of L, lies on Q.

Again, since Q_4 passes through xyz, we have $\delta' = d$, and, since it also passes through xya_5, we have

$$cz^2 + d\omega^2 + 2rz\omega = (\gamma' z + d\omega)\,\omega \quad \text{or} \quad cz + 2r\omega = \gamma'\omega.$$

But, at xya_5, $Cz + D\omega = 0$; whence $\gamma' = (2r/D - c/C)\,D$. Similarly

$$\beta' = (2q/D - b/B)\,D, \quad \alpha' = (2p/D - a/A)\,D;$$

therefore $L'/D \equiv (2p/D - a/A)\,x + (2q/D - b/B)\,y + (2r/D - c/C)\,z + d/D\,\omega$, and the condition that a_4, the pole of L', lies on Q is

$$\begin{vmatrix} a & h & g & p & 2p/D - a/A \\ h & b & f & q & 2q/D - b/B \\ g & f & c & r & 2r/D - c/C \\ p & q & r & d & d/D \\ 2p/D - a/A & 2q/D - b/B & 2r/D - c/C & d/D & \end{vmatrix} = 0.$$

But this condition is equivalent to the former one, since, by multiplying the fourth column of the former by 2/D, and subtracting the fifth column, for a new fifth column that determinant has the first four rows identical with those above, and the fifth row is

$$a/A \quad b/B \quad c/C \quad d/D \quad 2d/D^2.$$

Now, making a new fifth row by subtracting this fifth row from 2/D times the fourth row, we see that the conditions are absolutely identical.

Again, if the co-ordinates of a_4 be $(x_4, y_4, z_4, \omega_4)$, of a_5 be $(x_5, \ldots)$, the conditions that a_4a_5 passes through $a_1a_2a_3$, *i.e.*, through $x = y = z = 0$, are

$$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ x_4 & y_4 & z_4 & \omega_4 \\ x_5 & y_5 & z_5 & \omega_5 \end{Vmatrix} = 0, \quad \textit{i.e.}, \text{ that } y_4z_5 = y_5z_4,\ z_4x_5 = z_5x_4.$$

But, if a_4 be the pole of $L' = 0$, *i.e.*, of

$$(2p/D - a/A)\,x + (2q/D - b/B)\,y + (2r/D - c/C)\,z + d/D\,\omega = 0,$$

and (A), (B), (C), (D), F, G, H, P, Q, R denote the minors of a, b, c, d, f, g, h, p, q, r in Δ, we have

$$\Delta y_4 = K\,[H\alpha' + (B)\,\beta' + F\gamma' + Q\delta'], \quad \Delta z_4 = K\,[G\alpha' + F\beta' + (C)\,\gamma' + R\delta'],$$
$$\Delta x_4 = K\,[(A)\,\alpha' + H\beta' + G\gamma' + P\delta'],$$

where α', β', γ', δ' have the values given above, *i.e.*, that

$$x_4 : y_4 : z_4 = \begin{vmatrix} \alpha' & \beta' & \gamma' & \delta' \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix} : \begin{vmatrix} a & h & g & p \\ \alpha' & \beta' & \gamma' & \delta' \\ g & f & c & r \\ p & q & r & d \end{vmatrix} : \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ \alpha' & \beta' & \gamma' & \delta' \\ p & q & r & d \end{vmatrix}.$$

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Similarly $x_5 : y_5 : z_5 =$ similar determinants with $\alpha, \beta, \gamma, \delta$ in place of $\alpha', \beta', \gamma', \delta'$. Thus, rejecting common factors, we get

$$y_4 z_5 = \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ a/\mathrm{A} & b/\mathrm{B} & c/\mathrm{C} & d/\mathrm{D} \\ p & q & r & d \end{vmatrix} \Bigg| \begin{vmatrix} a & h & g & p \\ 2p/\mathrm{D}-a/\mathrm{A} & 2q/\mathrm{D}-b/\mathrm{B} & 2r/\mathrm{D}-c/\mathrm{C} & d/\mathrm{D} \\ g & f & c & r \\ p & q & r & d \end{vmatrix}.$$

In the first of these factors build up a new third row by subtracting the third row from 2/D times the fourth; this gives

$$(2p/\mathrm{D}-a/\mathrm{A})(2q/\mathrm{D}-b/\mathrm{B})(2r/\mathrm{D}-c/\mathrm{C})\,d/\mathrm{D},$$

and in the second factor build up a new second row by subtracting the second row from 2/D times the fourth row; this gives

$$a/\mathrm{A} \quad b/\mathrm{B} \quad c/\mathrm{C} \quad d/\mathrm{D}.$$

Hence $y_4 z_5 = y_5 z_4$, and, similarly, $z_4 x_5 = z_5 x_4$; therefore $a_4 a_5$ passes through $a_1 a_2 a_3$. The reciprocal proposition may be proved in a precisely similar way.

Generalizing for space of $n-1$ dimensions, take the equation of the hyper-quadric Q,

$$\mathrm{A}_1 a_1^2 + \mathrm{A}_2 a_2^2 + \ldots + \mathrm{A}_n a_n^2 + 2a_{12} a_1 a_2 + \ldots + 2a_{rs} a_r a_s + \ldots = 0 \quad (a_{rs} \equiv a_{sr}).$$

Take for the $(n+1)$-th hyper-plane (a_{n+1}) the equation

$$c_1 a_1 + \ldots + c_n a_n = 0;$$

then, if S_r be the hyper-quadric through the intersection of Q and a_r and through the n intersections of the other n "planes" $n-1$ at a time, we have

$$\mathrm{S}_r \equiv \mathrm{Q} - a_r (p_1 a_1 + \ldots + p_n a_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\mathrm{S}_{n+1} = \mathrm{Q} - (c_1 a_1 + \ldots + c_n a_n)(q_1 a_1 + \ldots + q_n a_n) \quad \ldots\ldots\ldots\ldots (2).$$

(2) gives that $q_1 = \mathrm{A}_1/c_1, \ldots, q = \mathrm{A}_s/c_s, \ldots$, yielding, as the condition that a_{n+1} lies on Q,

$$\begin{vmatrix} \mathrm{A}_1 & a_{12} & \ldots & a_{1n} & \mathrm{A}_1/c_1 \\ a_{12} & \mathrm{A}_2 & \ldots & & \mathrm{A}_2/c_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & \mathrm{A}_n & \mathrm{A}_n/c_n \\ a_1/c_1 & a_2/c_2 & \ldots & a_n/c_n & . \end{vmatrix} = 0.$$

Again, since S_r passes through $a_1 a_2 \ldots a_{r-1} a_{r+1} \ldots a_n$, therefore $p_r = \mathrm{A}_r$, and, since it also passes through $a_{n+1} a_2 a_3 \ldots a_{r-1} a_{r+1} \ldots a_n$,

$$\mathrm{A}_1 a_1^2 + \mathrm{A}_r a_r^2 + 2a_{1r} a_1 a_r = a_r (p_1 a_1 + p_r a_r) \quad \text{or} \quad \mathrm{A}_1 a_1 + (2a_{1r} - p_1 a_r) = 0,$$

which, coupled with $c_1 a_1 + c_r a_r = 0$, gives $p_1 = c_r (2a_{1r}/c_r - \mathrm{A}_1/c_1)$; so $p_2 = c_r (2a_{2r}/c_r - \mathrm{A}_2/c_2)$, with similar values for $p_3, \ldots, p_{r-1}, p_{r+1}, \ldots, p_n$, while $p_r = \mathrm{A}_r$. Hence the condition that $p_1 a_1 + \ldots + p_n a_n = 0$ shall touch Q, *i.e.*, that a_r lies on Q, is

$$\begin{vmatrix} \mathrm{A}_1 & \ldots & a_{1r} & \ldots & a_{1n} & 2\mathrm{A}_{1r}/c_r - \mathrm{A}_1/c_1 \\ a_{12} & \ldots & a_{2r} & \ldots & a_{2n} & 2a_{2r}/c_r - \mathrm{A}_2/c_2 \\ \ldots \quad \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \quad \ldots \\ a_{1r} & \ldots & \mathrm{A}_r & \ldots & a_{rn} & \mathrm{A}_r/c_r \\ \ldots \quad \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \quad . \\ a_{1n} & \ldots & a_{nr} & \ldots & \mathrm{A}_n & 2a_{nr}/c_r - \mathrm{A}_n/c_n \\ 2a_{1r}/c_r - \mathrm{A}_1/c_1 & \ldots & \mathrm{A}_r/c_r & \ldots & 2a_{nr}/c_r - \mathrm{A}_n/c_n & 0 \end{vmatrix} = 0.$$

But, if we build up a new $(n+1)$-th column by multiplying the r-th

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column by $2/c_r$ and subtracting the $(n+1)$-th column, we get for this column

$$A_1/c_1 \quad A_2/c_2 \quad \dots \quad A_n/c_n \quad 2A_r/c_r^2.$$

Similarly form a new $(n+1)$-th row by multiplying the r-th row by $2/c_r$ and subtracting the $(n+1)$-th row; we get for this new row

$$A_1/c_1 \quad A_2/c_2 \quad \dots \quad A_n/c_n \quad 0.$$

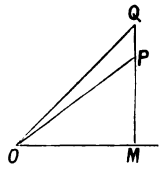
Thus the condition that a_{n+1} should lie on Q coincides with the condition that a_n should lie on Q.

15470. (SALUTATION.)—The sines of two angles are expressible in proper fractions the denominators of which differ by unity; and their cosines are inversely proportional to the respective denominators. The latter being given, find the several numerators.

Solution by W. SCRIMGEOUR, M.A., B.Sc.

Let $\angle MOP = A$, $\angle MOQ = B$,

$OP = a$, $OQ = a+1$, $MP = x$, $MQ = y$.



Then $\sin A = x/a$, $\sin B = y/(a+1)$,

$\cos A = OM/a$, $\cos B = OM/(a+1)$.

Hence the last condition is satisfied, the cosines being inversely proportional to the denominators.

$$OM^2 = a^2 - x^2 = (a+1)^2 - y^2;$$

therefore $y^2 - x^2 = 2a+1$. Let b, c be the factors of $2a+1$; therefore $(y+x)(y-x) = bc$. Let $y+x = b$, $y-x = c$; therefore $y = \frac{1}{2}(b+c)$, $x = \frac{1}{2}(b-c)$. Since $2a+1$ is odd, b and c are both odd; therefore x and y are integral. Give b, c all possible simultaneous values, and all the solutions should be obtained. Thus, if $a = 7$, $2a+1 = 15 = 3(5)$; therefore $y = 4$, $x = 1$; therefore $\sin A = \frac{1}{7}$, $\sin B = \frac{4}{8}$; $\cos A = \frac{1}{7}\sqrt{(48)}$, $\cos B = \frac{1}{8}\sqrt{(48)}$. This value of a also gives $y = 8$, $x = 7$.

[There is always the solution $x = a$, $y = a+1$, which gives

$$\cos A = \cos B = 0.]$$

$a = 22$ gives two other solutions, $x = 6$, $y = 9$; $x = 2$, $y = 7$.

Note.—There is possibly a general expression giving the *number* of values for x and y corresponding to a given value of a.

14862. (JOHN GRIFFITHS, M.A.)—If D be the foot of the perpendicular from the vertex A upon the base BC of a triangle ABC, whose orthocentre is P, and DA be produced to A′ so that $AA' = DP$, prove that the nine-point circle of ABC is a G.B. circle (generalized Brocard circle) of A′BC.

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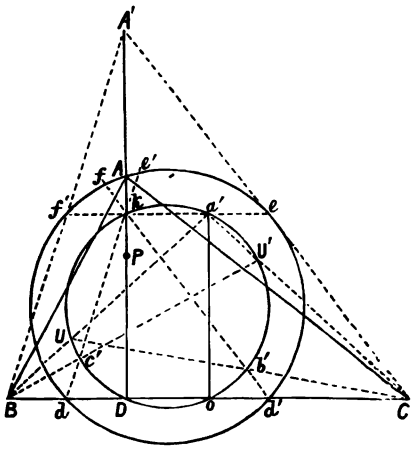
Solution by C. E. YOUNGMAN, M.A.

Let o, k be the mid-points of BC and DA′; then k is also mid-point of AP, and ok is a diameter of the circle. Find d, d', e, f', the mid-points of BD, DC, CA′, A′B; then $f'ke$ is parallel to BC, and kd, kd' are parallel to A′B, A′C or to oe, of'. Also $f'd$ and $d'e$ are both perpendicular to BC; therefore d, d', e, f lie on a circle. This circle is concentric with the other, since the diameters de and ok bisect each other. Let e', f be the points at which it cuts kd, kd' again.



Project o upon ke, kf, kd into the points a', b', c'; these will be vertices of a triangle inversely similar to A′BC, inscribed in the circle ok; and we have $a'e = f'k = Bd$, $b'f = d'k = Ce$; hence Ba' and Cb' are parallel to de and ef. Let them meet at U; then $\angle b'Ua' = \angle def = \angle dd'f = \angle BCA' = \angle b'c'a'$; therefore U lies on the circle ok, and also on the circle through BC touching CA′. Similarly, Bc' and Ca' meet at U′, lying on the circle ok and the circle through BC touching BA′. Hence

$$\angle UCA' = \angle UBC = \angle U'CB = \angle U'BA' = \omega' = 90° - \angle Ua'o$$
$$= 90° - \angle Uko = \angle Uok = \angle U'ok.$$

Therefore U, U′ are an isogonal pair in A′BC, and possess properties analogous to those of its Brocard points. We have also

$$\cot\omega' = Bo/oa' = BC/DA' = \cot A'BC + \cot A'CB.$$

Since parallels through d, e, f to A′B, BC, CA′ meet at k, parallels through C, A′, B to ef, fd, de must also meet; therefore A′U is parallel to fd; therefore CUA′ is a right angle; and similarly with A′U′B.

4991. (G. A. OGILVIE.)—Show (1) that there is only one point the polars of which with respect to four conics form a parallelogram, and (2) find the area of the parallelogram for any four conics.

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Note on the above Question by R. F. WHITEHEAD, B.A.

Suppose the polars of a point P with regard to two conics centres C_1, C_2 to be parallel, meeting in a point Q at infinity. Since C_1P, C_2P are the polars of Q, (1, 1) relations connect the lines C_1P, C_2P with Q, which traverses the line infinity, and therefore a (1, 1) relation connects them with each other. Hence P describes a conic through C_1C_2; obviously the conic will circumscribe the self-polar triangle. Hence, if we take the conics in pairs S_1, S_2 and S_3, S_4, there will be *four* points, the intersections of two such conics, whose polars form a parallelogram. Altogether there will be *twelve* such points.

15074. (Professor NEUBERG.)—Construire le centre et les asymptotes de l'hyperbole équilatère qui passe par deux points donnés A, B et qui touche une droite donnée d en un point donné C.

Solution by Professor SANJÁNA, M.A., *and* F. W. REEVES, B.A.

Since three points A, B, C on the curve are given, a fourth point, viz., the orthocentre O of the triangle ABC, is known to lie on it. The problem therefore reduces to the description of a conic passing through A, B, C, O, and touching d; this is easily effected. (See Besant's *Geometrical Conics*, § 250.)

14821. (Professor COCHEZ.)—Construire un triangle connaissant la hauteur et la bissectrice de l'angle A ainsi que la somme $b + c = m$.

Solution by C. E. YOUNGMAN, M.A.

Let ABC be the required triangle, with CA produced till AB′ = AB; AL the altitude; AD, AD′ the bisectors of the angle A; and M, N, X the mid-points of BB′, LD′, and BC. Then

$XM^2 = XL \cdot XD'$;

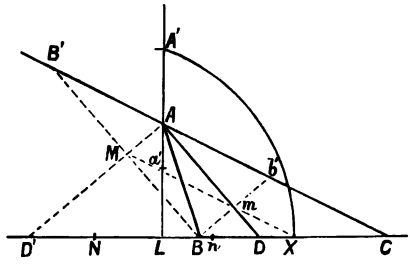
for $\angle XML$

$= \angle XMB + \angle BML$

$= \frac{1}{2}\angle A + \angle BAL$

$= \angle DAL = \angle LD'M$,

and XM touches the circle LD′M.



Now the triangle ALD is given; therefore also AD′ and N; and to

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find X we need only take LA′ along LA of the given length $XM = \frac{1}{2}(b+c)$, and then NX = NA′; and M, B, C follow in that order.

If $b-c$ were given instead of $b+c$, the same construction would do, modified as shown by the small letters. The problem is, in effect: To draw across an angle DD′B′ (or D′Db′), through a point A on its bisector, a line CB′ (or Cb′) of given length.

15435. (Professor NANSON.)—If t_r be the arithmetic mean of the r-th powers of n positive quantities which are not all equal, prove that t_1, $t_2^{\frac{1}{2}}$, $t_3^{\frac{1}{3}}$, $t_4^{\frac{1}{4}}$, ... are in ascending order of magnitude.

Solution by COLIN M. ROSS.

$$\frac{(A+A+\dots \text{ to } a \text{ terms})+(B+B+\dots \text{ to } b \text{ terms})+\dots}{a+b+\dots} > (A^aB^b\dots)^{1/(a+b+\dots)}$$

(Hall and Knight's *Higher Algebra*, § 253), *i.e.*,

$$[(aA+bB+\dots)/(a+b+\dots)]^{a+b+\dots} > A^aB^b\dots \quad \dots\dots\dots\dots(1),$$

In (1) substitute A^p, B^p, ... for a, b, ...; also A^{r-p}, B^{r-p}, ... for A, B, ..., where $r > p$; then

$$[(A^r+B^r+\dots)/(A^p+B^p+\dots)]^{A^p+B^p+\dots} > (A^{A^p}B^{B^p}\dots)^{r-p}\dots\dots(2),$$

In (1) again substitute A^p, B^p, ... for a, b, ... and A^{s-p}, B^{s-p}, ... for A, B, ..., where $p > s$; then

$$[(A^s+B^s+\dots)/(A^p+B^p+\dots)]^{A^p+B^p+\dots} > (A^{A^p}B^{B^p}\dots)^{s-p};$$

therefore

$$[(A^p+B^p+\dots)/(A^s+B^s+\dots)]^{A^p+B^p+\dots} < (A^{A^p}B^{B^p}\dots)^{p-s}\dots\dots(3),$$

Hence, as $r-p$ and $p-s$ are positive quantities, from (2) and (3),

$$[(A^r+B^r+\dots)/(A^p+B^p+\dots)]^{p-s} > [(A^p+B^p+\dots)/(A^s+B^s+\dots)]^{r-p}.$$

Put $s = 0$; then

$$[(A^r+B^r+\dots)/(A^p+B^p+\dots)]^p > [(A^p+B^p+\dots)/n]^{r-p};$$

therefore $$[(A^r+B^r+\dots)/n]^p > [(A^p+B^p+\dots)/n]^r,$$

i.e., $$t_r^p > (t_p)^r \quad \dots\dots\dots\dots(4).$$

Putting $r = 2, 3, 4, \dots$, and $p = 1, 2, 3, \dots$, successively in (4), it follows that $t_2^{\frac{1}{2}} > t_1$, $t_3^{\frac{1}{3}} > t_2^{\frac{1}{2}}$, $t_4^{\frac{1}{4}} > t_3^{\frac{1}{3}}$, and so on; therefore t_1, $t_2^{\frac{1}{2}}$, $t_3^{\frac{1}{3}}$, $t_4^{\frac{1}{4}}$, ... are in ascending order of magnitude.

15175. (R. KNOWLES.)—PQ is a chord of a conic at right angles to the transverse axis; the diameter through Q meets the normal at P in R. A circle is described with centre R and radius RP, cutting the conic in M, M′. Prove (1) that MM′ is a diameter of the circle; (2) that MM′ is parallel to the tangent at Q.

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Solutions (I.) *by* R. TUCKER, M.A., *and* LIONEL E. REAY, B.A.;
(II.) *by* F. W. REEVES, B.A.

(I.) The equation to the normal at P is

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2,$$

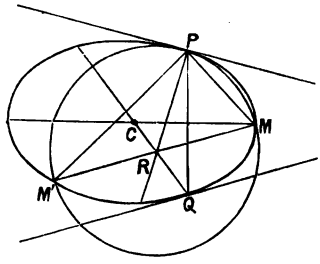
and to CQ is

$$ay\cos\phi + bx\sin\phi = 0;$$

hence at R

$$x = ap\cos\phi, \quad y = -bp\sin\phi,$$

where $p = (a^2-b^2)/(a^2+b^2)$.



Now the chord of the circle and the ellipse is equally inclined to the axis with the tangent at P (therefore parallel to the tangent at Q); hence the equation to the parallel chord through R is

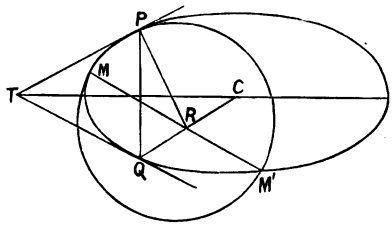
$$bx\cos\phi - ay\sin\phi = abp \quad \text{.....(i.).}$$

Let this chord cut the ellipse in the points ϕ', ϕ''; then

$$\tan P = -\frac{(b/a)[\cot\tfrac{1}{2}(\phi+\phi') - \cot\tfrac{1}{2}(\phi+\phi'')]}{1+(b^2/a^2)\cot\tfrac{1}{2}(\phi+\phi')\cot\tfrac{1}{2}(\phi+\phi'')} \quad \text{.....(ii.);}$$

but, from (i.), $\cos(\phi+\phi') = p = \cos(\phi+\phi'')$; hence (ii.), $\tan P = \infty$, *i.e.*, the angle P is a right angle, *i.e.*, the chord through R is a diameter of the circle.

(II.) Let the tangents at P, Q meet in T on the transverse axis; TP, TQ are equally inclined to the transverse axis. Because the circle meets the conic in four points two of which (since TP is tangential to both the circle and the conic) coincide at P, therefore MM′, TP are equally inclined to the axis; therefore MM′ is parallel to either TP or TQ. In the former case P must be the end of the minor axis; therefore MM′ is parallel to TQ; therefore CRQ bisects MM′; therefore, if R is not itself the mid-point of MM′, CRQ is perpendicular to MM′, and thus to the tangent at Q; therefore R is the mid-point of MM′; therefore MM′ is that diameter of the circle which is parallel to the tangent to the conic at Q.



15436. (R. TUCKER, M.A.)—In the figure to Question 15275 Ω, Ω are the positive and negative "B" points of ABC. Prove also that they are respectively the positive and negative "B" points of the triangles *abc*, *a′b′c′*. Find also the point of intersection of $\Omega\Omega_1$, $\Omega'\Omega_1'$, where Ω_1, Ω_1' are the other "B" points of *abc*, *a′b′c′*.

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Solution by the PROPOSER.

In Fig. 1 the co-ordinate of the positive "B" point of abc is

$$\frac{c\sin^2\theta}{\sin B}+\frac{ac^2\delta\sin(\theta+\omega)}{\lambda}=\frac{ac^2\sin^2\theta}{2\Delta}+\frac{ac^2}{\lambda}\,\frac{\sin^2\omega-\sin^2\theta}{\sin\omega}=\frac{ac^2\sin\omega}{\lambda}$$

= co-ordinate of Ω;

hence, &c. In Fig. 2, the co-ordinate of the negative "B" point of $a'b'c'$ is $\dfrac{b\sin^2\theta}{\sin C}+\dfrac{b^2a\delta\sin(\theta+\omega)}{\lambda}=\ldots=\dfrac{ab^2\sin\omega}{\lambda}$ = co-ordinate of Ω';

hence, &c., where $\delta=\sin(\omega-\theta)/\sin\omega$ is the modulus of similarity of

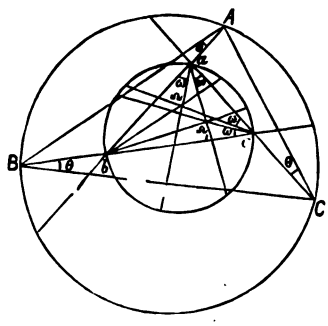

FIG. 1.

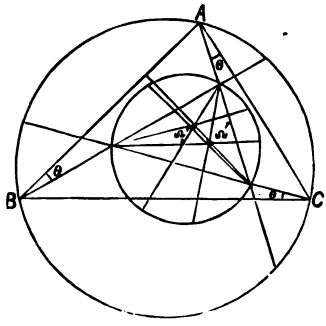

FIG. 2.

abc, $a'b'c'$. The notation above is that of my paper on the "Triplicate-Ratio Circle" (*Quart. Jour. of Math.*, Vol. XIX., No. 76).

Again, the co-ordinate of Ω_1 is

$$\frac{c\sin^2\theta}{\sin B}+\frac{a^2c\delta}{\lambda}\sin(B-O)=\frac{ac^2\sin^2\theta+a^2c\sin B'\sin O}{\lambda\sin\omega},$$

where O stands for $\omega-\theta$, and B′ for B−O; hence the equation to $\Omega\Omega_1$ is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ ac^2\sin^2\theta+a^2c\sin B'\sin O & \ldots & \ldots \\ ac^2 & ba^2 & cb^2 \end{vmatrix},$$

that is, $\Sigma ba\,(ca\sin A'-b^2\sin C')=0$. Similarly the equation to $\Omega'\Omega_1'$ is $\Sigma ca\,(ab\sin A'-c^2\sin B')=0$, and these two lines intersect in

$$\frac{\alpha}{\left[bc\,(b^2c^2\sin^2A'+a^4\sin^2B'+a^4\sin^2C')\right.}=\ldots=\ldots.$$
$$\left.-(a^5\sin B'\sin C'+abc^4\sin C'\sin A'+ab^4c\sin A'\sin B')\right]$$

(*Cf.* Question 15275.)

15463. (ROBT. W. D. CHRISTIE.)—Solve $X^2-44537Y=-1$, and generalize your solution.

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by the* PROPOSER.

(I.) The Question, stated generally, is the same thing as solving $X^2+1=m.Y$, where m is given. And, since m divides into (X^2+1^2),

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it follows that m itself $= a^2+b^2$. Now, when a, b in $m = a^2+b^2$ are known, then the congruence $X^2+1 \equiv 0 \pmod{m}$ can readily be solved as follows. We have $(a \pm \lambda . m)^2 \equiv a^2 \equiv -b^2 \pmod{m}$. Now choose λ so that $a \pm \lambda . m = A . b$ (a multiple of b) (this is always possible). Hence $A^2 . b^2 \equiv -b^2$, and $A^2 \equiv -1 \pmod{m}$, provided b is prime to m. This gives $A^2+1 \equiv 0 \pmod{m}$, whence $(A^2+1)/m = B$ (suppose), and $X = \pm A$, $Y = B$ gives two solutions; and the general solution is

$$X = A \pm \lambda' . m, \quad Y = (X^2+1)/m.$$

Ex.--Given $m = 44537$. Here $m = 211^2+4^2 = a^2+b^2$; and $(a+m)$ is divisible by $b = 4$. Hence

$$(211+44537)^2 = (a+m)^2 = 44748^2 = (4 . 11187)^2 \equiv -b^2 = -4^2 \pmod{m};$$

therefore $A = 11187 = X$, $Y = (A^2+1)/m = 2810$; therefore

$$11187^2+1 = 44537 . 2810.$$

Note.—The present writer's *Tables of Quadratic Partitions* (now passing through the press) give the values of a, b in $p = a^2+b^2$ for all primes $< 100{,}000$; these suffice for solving the congruence $X^2+1 \equiv 0 \pmod{p}$ up to same limit. The present writer has also prepared a table (which it is hoped shortly to publish) giving the actual solutions of the congruence for all primes 10,000, and also of many 5-figure primes. For a description of this table see the writer's paper "On High Primes $(4\varpi+1)$, $(6\varpi+1)$, and Factorizations" in the *Quarterly Journal of Pure and Applied Mathematics*, Vol. xxxv., pp. 10, 11.

(II.) We have $44537 = 211^2+4^2$ obvious by using a table of squares or thus:—Squares end in the digits 0, 1, 4, 5, 6, 9 only. Of these endings $6+1$ give 7. Now inspect the table of squares. For this prime or by the general method given in the solution to a previous Question we easily get (*v.* "Factorisation of $r^{6m}+r^{2m}+2 = pq$") $\pm r^m = \pm 11187 \bmod 44537$; therefore $(\pm r^m)^2+1 = 44537 . 2810$ and $44537 . 24973$, and generally $(pq \pm r^m)^2+1 = p(pq^2 \pm 2qr^m+S)$, where p is given, q is any integer whatever, $\pm r^m$ and S are obtained (see the Proposer's paper on "Primitive Roots applied to the Pellian Equation," read before the Liverpool Mathematical Society, December 9th, 1903).

The question is a special case of the general form

$$(4a^2-a+4)(12a^2-7a+13) - \{(4a-1)^2+4^2\}(3a^2-a+3) = 1.$$

Hence $X = 4a^2-a+4$ or $12a^2-7a+13$, $Y = a^2+1$ or $9a^2-6a+10$, for all integral values of a. Here $a = 53$.

An infinite number of integral values of X and Y are got from

$$X \equiv \{(4a-1)^2+4^2\}\,Q + 4a^2-a+4$$

or

$$\equiv \{(4a-1)^2+4^2\}\,Q + 12a^2-7a+13$$

for all values of Q, with corresponding values of Y.

15189 (*re-stated*). (R. W. D. Christie.)—Let $N = T_a+T_b+T_c$ (three triangulars); then $2N+1 = p^2+q^2+r^2+s^2$ integral, where p, q, r, s are functions of a, b, c. [*Cf.* Vol. i., New Series, p. 94.]

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Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by the* PROPOSER.

(I.) T_n denotes $\frac{1}{2}n(n-1)$. Hence

$$\begin{aligned} 2N+1 &= 2(T_a+T_b+T_c)+1 \\ &= a(a-1)+b(b-1)+c(c-1)+1 \\ &= (a-1)^2+(b-1)^2+(c-1)^2+(a+b+c-2) \\ &= \text{sum of 4 squares if } (a+b+c-2)=z^2. \end{aligned}$$

As there are three quantities a, b, c at disposal, the condition N = any integer can easily be satisfied.

(II.) We have $8N+3 = (2a+1)^2+8N_1+2$ necessarily,

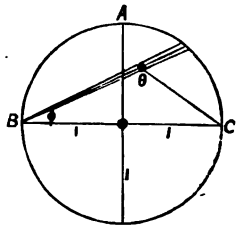
$$\begin{aligned} 8N_1+2 &= (2b+1)^2+8N_2+1 \quad ,, \\ 8N_2+1 &= (2c+1)^2+8N_3 \quad ,, \\ 8N_3 &= (2d+1)^2-1 \quad ,, \end{aligned}$$

$$\begin{aligned} \therefore\ 8N+4 &\equiv (2a+1)^2+(2b+1)^2+(2c+1)^2+(2d+1)^2 \text{ all odd,} \\ &\equiv (a+b+c-d)^2+(b+c+d-a)^2+(c+d+a-b)^2+(d+a+b-c)^2 \\ &\qquad \text{all even;} \\ \therefore\ 2N+1 &\equiv p^2+q^2+r^2+s^2 \text{ integral.} \end{aligned}$$

15447. (Professor M. W. CROFTON, F.R.S.)—Show that the mean value of the angle (θ) which the diameter of a semi-circle subtends at all points of its area is $M(\theta) = \frac{1}{2}\pi + 2/\pi$.

Solution by R. CHARTRES.

$$\begin{aligned} M(\theta) &= \pi - 2M(\phi) \\ &= \pi - 2\int_0^{\frac{1}{2}\pi} \phi\,[\tfrac{1}{2}(2\cos\phi)^2\,d\phi]/\tfrac{1}{2}\pi \\ &= \pi - (\tfrac{1}{2}\pi - 2/\pi) \\ &= \tfrac{1}{2}\pi + 2/\pi. \end{aligned}$$



It is remarkable that, if the random point was confined to the triangle BAC, the mean of the supplement of θ would

$$= \log 2.$$

15487. (A. M. NESBITT, M.A.)—Resolve into factors

$$\begin{vmatrix} a & . & . & l & . & . \\ . & b & . & . & m & . \\ . & . & c & . & . & n \\ g & . & . & d & . & . \\ . & h & . & . & e & . \\ . & . & k & . & . & f \end{vmatrix}$$

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Solutions (I.) *by* R. F. WHITEHEAD, B.A.; (II.) *by* Prof. SANJÁNA, M.A.; (III.) *by* W. F. S. CHURCHILL.

(I.) Expanding by Laplace's method,

$$\text{determinant} = (ad-gl)\begin{vmatrix} b & . & m & . \\ . & c & . & n \\ h & . & e & . \\ . & k & . & f \end{vmatrix} = (ad-gl)(be-hm)(cf-kn).$$

(II.) If $a = gl/d$, we have the first column equivalent to the fourth multiplied by g/d; so that the determinant is zero. Hence $(ad-gl)$ is a factor; so also are $(be-hm)$ and $(cf-kn)$. There is no numerical factor, as the chief diagonal constituent is *abcdef*.

(III.) $$\Delta^2 = \begin{vmatrix} a & . & . & l & . & . \\ . & b & . & . & m & . \\ . & . & c & . & . & n \\ g & . & . & d & . & . \\ . & h & . & . & e & . \\ . & . & k & . & . & f \end{vmatrix} \begin{vmatrix} . & . & l & . & . & -a \\ . & -m & . & . & b & . \\ n & . & . & -c & . & . \\ . & . & d & . & . & -g \\ . & -e & . & . & h & . \\ f & . & . & -k & . & . \end{vmatrix}$$

$$= \begin{vmatrix} . & . & A & . & . & B \\ . & . & . & . & C & . \\ -A & . & . & D & . & . \\ . & . & -D & . & . & E \\ . & -C & . & . & . & . \\ -B & . & . & -E & . & . \end{vmatrix}$$

where $A = an-lc$, $B = af-lk$, $C = hm-be$, $D = dc-ng$, $E = fg-dk$

$$= C^2\begin{vmatrix} . & A & . & B \\ -A & . & D & . \\ . & -D & . & E \\ -B & . & -E & . \end{vmatrix} = C^2(AE+BD)^2,$$

$$AE+BD = (ad-lg)(fc-nk);$$

therefore $$\Delta = \pm C(AE+BD) = \pm(hm-be)(ad-lg)(fc-nk)$$
$$= (ad-lg)(be-hm)(fc-nk).$$

Note.—This method of evaluating determinants by multiplying by the determinant reversed with the signs of alternate columns changed is of great use, and does not seem to be specially mentioned in text-books.

15359. (ROBT. W. D. CHRISTIE.)—Without the use of the method of continued fractions, solve the equation $X^2-149Y^2 = 1$, and generalize your method.

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Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.; (II.) *by* R. F. DAVIS, M.A.; (III.) *by* A. H. BELL.

(I.) Since $13^2-149\,.\,1^2=+20$ and $12^2-149\,.\,1^2=-5$, therefore

$$-4=(13^2-149\,.\,1^2)/(12^2-149\,.\,1^2)=t_1^2-149\,.\,u_1^2, \text{ suppose.}$$

The values of t_1, u_1 may be found by the method of conformal division (see Col. Cunningham's paper "On the Connexion of Quadratic Forms" in *Proc. Lond. Math. Soc.*, Vol. XXVIII., 1896, p. 289), whence

$$t_1=\tfrac{1}{5}(13\,.\,12+149\,.\,1\,.\,1)=61, \quad u_1=\tfrac{1}{5}(13\,.\,1+12\,.\,1)=5.$$

Thus $61^2-149\,.\,5^2=-4$. Now square this; then

$$(61^2+149\,.\,5^2)^2-149\,(2\,.\,61\,.\,5)^2=7446^2-149\,.\,610^2=+16,$$

giving $$3723^2-149\,.\,305^2=+4.$$

Hence $-1=(3723^2-149\,.\,305^2)/(61^2-149\,.\,5^2)=t_2^2-149\,.\,u_2^2$ suppose; and, by conformal division,

$$t_2=\tfrac{1}{4}(61\,.\,3723+149\,.\,5\,.\,305)=113582,$$
$$u_2=\tfrac{1}{4}(61\,.\,305+5\,.\,3723)=9305;$$

therefore $113582^2-149\,.\,305^2=-1$. Now square this; then

$$(113582^2+149\,.\,305^2)^2-149\,(2\,.\,305\,.\,113582)^2=+1,$$

or $$25801741449^2-149\,.\,2113761020^2=+1.$$

To generalize the above, *i.e.*, to find $X^2-D\,.\,Y^2=+1$ (D being given), compute the quantities $x_1^2-D\,.\,y_1^2=a_1$, $x_2^2-D\,.\,y_2^2=a_2$, ... for any chosen values of (x_1, y_1), (x_2, y_2), ..., until two results a_m, a_n are obtained such that $a_n=k\,.\,a_m$. Then (in some cases not easily specified) the value of $k=a_n/a_m$ can be obtained in the form $t_1^2-D\,.\,u_1^2=k$, by conformal division.

By further trials, another partition of k may certainly be found, say, $\pm k=t'^2_1-Du'^2_1$. When this is found the partition $X'^2-D\,.\,Y'^2=\pm 1$ can always be found by conformal division; and, if the result is -1 (*i.e.*, if the values of k above be of opposite sign), then $X=X'^2+D\,.\,Y'^2$, $Y=2X'Y'$ will satisfy $X^2-DY^2=+1$.

(II.) If $x=\alpha$, $y=\beta$ satisfy the equation $x^2-Ny^2=-1$, it may easily be shown that $x=2N\beta^2-1$, $y=2\alpha\beta$ satisfy the equation $x^2-Ny^2=1$. For instance, $x=18$, $y=5$ satisfy the equation $x^2-13y^2=-1$; therefore $x=649$, $y=180$ must satisfy the equation $x^2-13y^2=1$. Let us endeavour to solve $x^2-149y^2=-1$ as a preliminary step to solving the equation $x^2-149y^2=1$.

$$149y^2=1+x^2=(\text{say})\,(7u-10v)^2+(10u+7v)=149\,(u^2+v^2).$$

Hence we require to make $7u-10v=1$ and $u^2+v^2=y^2$. The first condition is satisfied by taking $u=10t+3$, $v=7t+2$; and the second condition gives $(10t+3)^2+(7t+2)^2=\square$, whence $t=762$ (by the help of $t=-\frac{1}{2}$, for which $10t+3:7t+2=4:3$). Thus $u=7623$, $v=5336$; and $x=113{,}582$, $y=9{,}305$ satisfy the equation $x^2-149y^2=-1$. Therefore $x=2\,.\,149\,.\,9{,}305^2-1$ and $y=2\,.\,113{,}582\,.\,9{,}305$ will satisfy the equation $x^2-149y^2=1$.

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(III.) Let us first consider the general equation $x^2 - Ny^2 = \pm 1$, and assume

$$Ny^2 + 1 = x^2 = (Nmy/n - 1)^2 \quad \text{and} \quad y = 2mn/(Nm^2 - n^2) \; [x = (Nmy/n) \pm 1] \quad \ldots\ldots\ldots(1).$$

To have integral values of y, $Nm^2 - n^2$ must be found a factor of $2mn$; when it is one, by transposition we have a solution of the proposed equation for the condition -1. Now

$$Nm^2 = n^2 + d = (n+1)^2 - d' \quad \ldots\ldots\ldots(2),$$

by taking successively $m = 1, 2, 3, \ldots$, discloses the law of development, and it will be found that the smallest values of d return at equal intervals, alternately changing signs, but the corresponding values of m, n have increased in a uniform arithmetical ratio. The smallest values of the d's occur at a change of signs in the series, which we call convergent differences, C.D. This series is one of two differences, and to develop the three series of m, n, d require three terms of (2), and they have these relations with each other. Let L and F be written for last term of a series and first term of next, and A.S. for an arithmetical series, D for difference.

No. of term.	Diff.	0,	1,	.....,	Diff.,	L. and F.
A.S. m =	m'''	m +	m'	.....	m^0,	m''
A.S. n =	n'''	n +	n'	.....	n^0,	n''
C.D. =	$(\pm d''')$	$(\pm d)$,	$(\mp d')$	.....	$(\mp d^0)$,	$(\pm d'') : (\pm d^{iv})$
1st Diff. =	$(\mp d') - (\pm d)$	D_1,	D'_1	.....	$D_1{}^0$	$D_1'' : D_1{}^{iv}$
2nd Diff. =	$(\pm 2d'')$	$(\pm D_2)$,	D'_2	.....	$(\mp D_2{}^0)$,	$(\mp 2d^0)$

$[D_2{}^0 = 3d^0 - d''$, where d''' and d^0 belong to two consecutive diff. terms] $\ldots\ldots\ldots(3)$.

The difference term supplies the ratios for extending a triple series; this series terminates when the series of d's changes signs; the last term is the first term of the next series, and the preceding term provides the ratios for extending it, &c. Where the same C.D. with a change of sign appears in the next series we find the middle of the development, and it can be proved that, if two such terms are represented by

m^0, m, then $q = m^2 + m^{0^2}$; if by n^0, n, then $p = mn + m^0 n^0$, the end of cycle; if by $+d^0$, $-d^0$, then 1 ....(4).

It is observed that m''', n''' are the common differences for the m', n' of the first term in (3) respectively; also the sign of d''' in this difference term must be opposite to that of the first term; and twice this d''' is D_2. Also in developing the triple series as shown in (3), the next to the last term becomes the difference term for the next series, by supplying an intermediate $(D_2{}^0)$; the initial terms 0 and 1 are omitted, and the second term becomes the first term of the new series. This initial of the beginning of the development of a new series in the d's explains how the formula is obtained for this intermediate $(D_2{}^0)$, and why we immediately extend the triple series by taking the last term for the first term of the next series. In commencing a triple series by means of (2) it is only necessary to select such initial terms as have the conditions given in (3), and there need be no hesitancy then in arriving at the true and final results sought.

The solution of the general indeterminate equation of the second degree as given above in (1) represents the method used by Diophantus, and

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ordinarily these formulæ are as difficult to determine as the original equation. By abridging the development still more than as here shown, it is possible to arrive at the values of x, y much more easily than by any other known method.

Now consider the solution of $X^2-149Y^2=1$, in whole numbers without the aid of continued fractions.—We have, by (2),

$$\begin{array}{lll} \text{N.}\,m^2 & n^2+d & n'^2-d' \\ 149\,.\,1^2 = & 12^2+5 = & 13^2-20, \\ 149\,.\,2^2 = & \ldots = & 25^2-29, \\ 149\,.\,3^2 = & \ldots = & 37^2-28. \\ 149\,.\,4^2 = & \ldots = & 49^2-17, \\ 149\,.\,5^2 = & 61^2+4 \ldots\ldots, & \end{array}$$

....(5).

On account of the small d's we choose this last for the difference term, and the $12+5=0$ term, their difference, shows that $49-17$ must be the first term to commence the triple series (3); then

No.	Dif.	0	1	2	3	4	5	Dif.	7 & 1
Series m,	5 =	1,	+ 4,	...,	...,	...,	...,	29,	34 :
,, n,	61 =	12,	+49,	...,	...,	...,	...,	354,	415 :
,, d,	+4	+ 5,	−17,	−31,	−37,	−35,	−25,	− 7,	+19 :
	D_1 =	−22,	−14,	− 6,	+ 2,	+10,	+18,	+26,	+ 1 :
	D_2, 2 . 4,	+ 8,	+ 8,	+ 8,	+ 8,	+ 8,	+8 : (−25) :		−14 :

SECOND SERIES.

2	3
...,	92,
...,	1123,
+20,	+7,
−13,	

by (3), $(-25) = -3\times 7-4$

.....(6).

The series of d ends at the seventh term, and of the arithmetical series m, n, whose common differences are 5, 61, we need only the difference and the seventh terms; continuing the d series, until finding the same d, 7 with the sign changed shows the middle of the development (4); then the formulæ give

$m = q = 92^2+29^2 = 9305$, $n = p = 92\,.\,1123+29\,.\,354 = 113582$, $d = +1$

....(7).

These values $n=x$, $m=y$ satisfy the equation $x^2-149y^2=-1$. When these values of m, n are substituted in formulæ (1) we have the solution of the proposed equation

$$y = 2mn = 2\,.\,9305\,.\,113582 = 2113761020,$$
$$x = 2\,.\,149\,.\,9305^2-1 = 25801741449.$$

The triple series could have commenced with $25-29$ for difference term; then (3) A.S. m, $2=1+1$, A.S. n, $25=13+12$, and series d is -29, -20, $+5$, $D=-58$, and developed by the above method will soon merge into the development as given. When (D^0) causes a change of signs immediately the series ends, and a new one commences, which requires a new (D^0), but is used but once in commencing a series.

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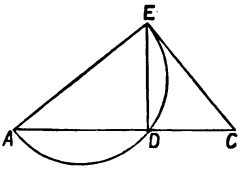
Geometrical Note.—To construct a right-angled triangle with the three sides in geometrical progression, the hypotenuse being given.

By CONSTANCE I. MARKS, B.A.

Let AC be the given hypotenuse. Divide AC at D as in Euclid II. 11, so that

$$AC \cdot DC = AD^2.$$



From D draw DE perpendicular to AC, and with C as centre and radius equal to AD describe a circle cutting DE at E. Join AE and on AE as diameter describe the semi-circle ADE. AEC is the required triangle. For $AC \cdot DC = AD^2 = CE^2$; therefore CE is a tangent to the circle ADE, and, since AE is a diameter, therefore the angle AEC is a right angle; hence $AE^2 = AC \cdot AD = AC \cdot CE$.

15484. (Lt.-Col. J. HICKMAN.)—Smith takes a number in a small sweepstakes of 100 chances at 2s. 6d. each on a race in which 12 horses are entered. After drawing he forgets the name of the horse he drew, remembering only that it began with the letter *N*. When the race has been run he learns that the name of the winner begins with an *N*. The only further light he has as to the probability of his having won is that he believes $\frac{1}{10}$ of all racers bear names beginning with *N*. Proceeding on that assumption, find the precise mathematical value of his chance if the winner is to take $\frac{4}{5}$ of the stakes.

Solutions (I.) *by* D. BIDDLE; (II.) *by* V. DANIEL, B.Sc., *and* E. F. TIPPLE.

(I.) Col. Hickman has done well to call attention, through the medium of this Question, the details of which are familiar to those who entered for the recent "*Times* Competition," to the unsatisfactory answer given to the example in the *Encyclopædia Britannica* (Vol. XIX., p. 774), by reference to which the competitors were evidently intended to solve it. In the said example, the winning horse, like that drawn, is *black* (instead of being classed under the initial of its name); otherwise, so far as the chance of identity is concerned, the two are on all fours. The answer to the example, if applied to the present (or "Competition") question, would make the chance to be $1/(1+\frac{1}{10})$ or $\frac{10}{11}$, and, as $\frac{4}{5}$ of the stakes = £10, the required expectation is £6. 9s. $0\frac{10}{11}$d. according to the *Encyclopædia*. The method therein adopted is crude indeed, and I venture (though with some fear and trembling) to submit the following instead.

That one of the 12 horses has a name beginning with *N*, like that drawn, is certain; for the winner's name begins with *N*. The several chances that, of the 11 remaining horses, there are 0, 1, 2, 3, 4, ..., 11

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with names beginning with N are given by the successive terms in the expansion of $(\frac{1}{10}\frac{9}{0}+\frac{1}{10})^{11}$, namely, $\frac{19^{11}}{20^{11}}$, $\frac{11\,.\,19^{10}}{20^{11}}$, $\frac{55\,.\,19^{9}}{20^{11}}$, $\frac{165\,.\,19^{8}}{20^{11}}$, $\frac{330\,.\,19^{7}}{20^{11}}$, $\frac{462\,.\,19^{6}}{20^{11}}$, $\frac{462\,.\,19^{5}}{20^{11}}$, $\frac{330\,.\,19^{4}}{20^{11}}$, $\frac{165\,.\,19^{3}}{20^{11}}$, $\frac{55\,.\,19^{2}}{20^{11}}$, $\frac{11\,.\,19}{20^{11}}$, $\frac{1}{20^{11}}$. Here the common denominator is 204,800,000,000,000, whilst the several numerators are as follows :—

(1)	116490258898219,	(7)	1143957738,
(2)	67441728835811,	(8)	43005930,
(3)	17747823377845,	(9)	1131735,
(4)	2802287901765,	(10)	19855,
(5)	294977673870,	(11)	209,
(6)	21735197022,	(12)	1.

The numbers in parentheses not only represent the successive terms, but the number of horses having the particular initial N in the field, including the winner. These numerators must accordingly be divided severally and respectively by 1, 2, 3, 4, ..., yielding

116490258898219·000
33720864417905·500
5915941125948·333′
700571975441·250
58995534774·000
3622532837·000
163422534·000
5375741·250
125748·333′
1985·500
19·000
·08′3′

These numerators divided by the common denominator represent Smith's several chances, each being a combination of the chance that the horses in the field and initialed N are 1, 2, 3, ..., 12, with the particular chance that Smith thereupon wins. Their sum gives his total chance, namely,

$$\frac{156890423411153\cdot250}{204800000000000\cdot000}=\frac{627561693644613}{819200000000000},$$

or, instead of less than $\frac{3}{4}$, rather more than $\frac{4}{5}$; whilst his expectation, instead of being only £6. 9s. $0\frac{1}{2}\frac{1}{2}$d., is £7. 13s. $2\frac{1}{4}$d.

(II.) However small the frequency of the winner's initial, it must of necessity occur once, and from Smith's discovery nothing can be argued as to the relative probabilities of truth of 12 possible hypotheses. Hence we are concerned only with the *a priori* probabilities of the 11 other initials including 0, 1, 2, ..., 11 N's. These are the several terms of the expansion $(\frac{1}{10}\frac{9}{0}+\frac{1}{10})^{11}$. His expectation on the r-th of the hypotheses is $S\mu_r/r$; therefore total expectation

$$=S\Sigma\mu_r/r=20S\int_0^{\frac{1}{10}}(\tfrac{1}{10}\tfrac{9}{0}+x)^{11}dx=20S\tfrac{1}{12}[1-(\tfrac{1}{10}\tfrac{9}{0})^{12}]=(\cdot76607)\,S$$
$$=£7\cdot6607.$$

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Had a group of two or more, discovering they possessed the winner's initial, met and desired to calculate their expectation, it seems to me that the *a priori* probabilities of the 12 hypotheses would give place to *a posteriori* probabilities dictated by the knowledge of 1 at least, 2 at least, ..., N's among the 11 names which exclude the winner.

15450. (Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Find the product of the sum and difference of the series

$1 + 6 + 21 + 52 + 105 + \ldots$ to n terms, $3 + 14 + 39 + 84 + 155 + \ldots$ to n terms.

Additional Solution by H. CHENNAKESAVA IYENGAR.

Adding the two series, we have 4, 20, 60, 136, 260, ... ;

$$2 \cdot 2 + 4 \cdot 5 + 6 \cdot 10 + 8 \cdot 17 + 10 \cdot 26 + \ldots .$$

Now, in the series 2, 5, 10, 17, 26, ... the nth term $= n^2 + 1$; therefore $T_n = 2n(n^2+1) = 2n^3 + 2n$; therefore

$$S_n = 2\Sigma n^3 + 2\Sigma n = \tfrac{1}{2}n^2(n+1)^2 + n(n+1) = \tfrac{1}{2}n(n+1)(n^2+n+2).$$

Subtracting the first series from the second, we have 2, 8, 18, 32, 50, ... ; $2 \cdot 1 + 2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + \ldots$; therefore $T'_n = 2n^2$; therefore

$$S'_n = 2\Sigma n^2 = \tfrac{1}{3}n(n+1)(2n+1);$$

therefore $$S_n \cdot S'_n = \tfrac{1}{6}n^2(n+1)^2(n^2+n+2)(2n+1).$$

15441. (Professor NANSON).—Three triangles Δ_1, Δ_2, Δ_3 are inscribed in a conic. If Δ_2, Δ_3 are co-polar with Δ_1, then so is the triangle formed by the joins of corresponding vertices of Δ_2, Δ_3.

Solution by A. R. RICHARDSON *and* A. M. NESBITT, M.A.

Refer the conic to Δ_1 as triangle of reference.

$$S \equiv 2fyz + 2gzx + 2hxy = 0.$$

Eliminating x, y, z in turn from $hy + gz = 0$, $hx + fz = 0$, $gx + fy = 0$, we get $Fx = Gy = Hz$ as equations to the lines joining the vertices of Δ_1 with its polar triangle ; therefore (1/F, 1/G, 1/H) are co-ordinates of the pole of Δ. This form will hold whatever triangle be taken as the triangle of reference. Therefore the pole of Δ_2 is (1/F′, 1/G′, 1/H′); of Δ_3, (1/F″, 1/G″, 1/H″), where F′, F″, ... are linear functions of F. Now the triangle formed by joining the vertices of Δ_2, Δ_3 has co-ordinates of its vertices linear functions of those of Δ_2, Δ_3, say $k\Delta_2 + l\Delta_3$; hence the pole of ${}_3\Delta_2$ is $\left(\frac{1}{kF' + lF''}, \frac{1}{kG' + lG''}, \frac{1}{kH' + lH''}\right)$; but, since Δ_2, Δ_3 are copolar with Δ_1, therefore $F : F' : F'' = G : G' : G'' = H : H' : H''$; therefore ${}_3\Delta_2$ is also copolar with Δ_1.

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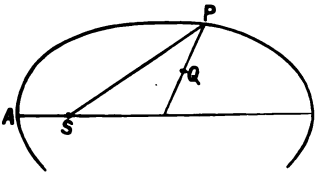
15478. (Professor NANSON.)—If r, θ are the focal polar co-ordinates of the foot of a normal from r', θ' to a conic, prove that $r_1r_2r_3r_4 = b^2e^{-2}r'^2$, $\theta_1+\theta_2+\theta_3+\theta_4 = 2\theta' + (2n+1)\pi$.

Solution by Professor SANJÁNA, M.A., *and* COLIN M. ROSS.

The first part is Ex. 77, chap. x., in Smith's *Conic Sections*, k^2 being there put for b^2e^{-2}.

For the second result, let Q be r', θ'; let P have $\angle ASP = \alpha$, and let PQ be normal. Then α is found from



$$le\sin\alpha/(1+e\cos\alpha)\,r' = e\sin\theta' + \sin(\theta'-\alpha)$$

(Smith, *Conic Sections*, § 165, Ex. 5). Let t denote $\tan\frac{1}{2}\alpha$, and we get the equation

$$\frac{le}{r'}\frac{2t}{1+t^2} = e\sin\theta' + e^2\sin\theta'\frac{1-t^2}{1+t^2} + \sin\theta'\frac{1-t^2}{1+t^2} - \cos\theta'\frac{2t}{1+t^2} + e\sin\theta'\frac{(1-t^2)^2}{(1+t^2)^2} - e\cos\theta'\frac{2t(1-t^2)}{(1+t^2)^2},$$

which reduces to

$$e\sin\theta'.\,2(1+t^4) + (1+e^2)\sin\theta'(1-t^4) - 2t(1+t^2)(\cos\theta' + le/r') - 2t(1-t^2)e\cos\theta' = 0.$$

Hence $t^4(e-1)^2\sin\theta' + 2t^3(\cos\theta' + le/r' - e\cos\theta') + 2t(\cos\theta' + le/r' + e\cos\theta') - (e+1)^2\sin\theta' = 0$;

so that, if the four values of α be θ_1, θ_2, θ_3, θ_4, we get

$$\Sigma\tan\tfrac{1}{2}\theta_1 = -\frac{2(\cos\theta' + le/r' - e\cos\theta')}{(e-1)^2\sin\theta'}, \quad \Sigma\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2 = 0,$$

$$\Sigma\tan\tfrac{1}{2}\theta_1\tan\tfrac{1}{2}\theta_2\tan\tfrac{1}{2}\theta_3 = -\frac{2(\cos\theta' + le/r' + e\cos\theta')}{(e-1)^2\sin\theta'}, \quad \Pi\tan\tfrac{1}{2}\theta = -\frac{(e+1)^2}{(e-1)^2}.$$

Therefore

$$\tan\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2} = \frac{-\frac{2(\cos\theta' + le/r' - e\cos\theta')}{(e-1)^2\sin\theta'} + \frac{2(\cos\theta' + le/r' + e\cos\theta')}{(e-1)^2\sin\theta'}}{1 - 0 + \frac{-(e+1)^2}{(e-1)^2}}$$

$$= \frac{4e\cos\theta'}{-4e\sin\theta'} = -\cot\theta' = \tan\left\{\theta' + (2n+1)\tfrac{1}{2}\pi\right\};$$

so that $$\theta_1+\theta_2+\theta_3+\theta_4 = 2\theta' + (2n+1)\pi.$$

15472. (Professor NEUBERG.)—On marque sur les côtés BC, CA, AB d'un triangle ABC les points A', B', C' tels que BA'/A'C = CB'/B'A = AC'/C'B = K. On sait que les droites AA', BB', CC' sont équipollentes aux côtés d'un triangle A''B''C''. L'aire de ce triangle est minimum lorsque K = 1.

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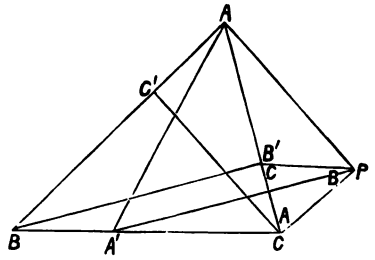
Solutions (I.) *by the* PROPOSER; (II.) *by* R. F. DAVIS, M.A.; (III.) *by* D. BIDDLE.

(I.) On peut projeter le triangle ABC orthogonalement sur un plan suivant un triangle équilatéral. Alors les lignes AA′, BB′, CC′ se projettent suivant trois droites égales qui passent par un minimum lorsque A′, B′, C′ sont les milieux de BC, CA, AB, ...

(II.) Draw A′P equal and parallel to BB′; then B′P will be equal and parallel to BA′. Join CP, AP. Since

$$CB' : B'P = CB' : BA' = CA : BC,$$

and $\angle CB'P = \angle ACB$, therefore the triangle CPB′ is similar to the triangle ABC. Thus PC is equal and parallel to AC′, and consequently PA is equal and parallel to CC′. The triangle AA′P has therefore its sides equal and parallel to AA′, BB′, CC′. The quadrilateral AA′CP is equal to the triangle ABC; for $\triangle APC = \triangle AC'C = \triangle AA'B$. The triangle AA′P is least when the triangle A′CP is greatest, that is, when A′C . CP is greatest, or when $[K/(K+1)][1/(K+1)]$ is greatest, which happens when $K = 1$.



(III.) Without altering the result we may take

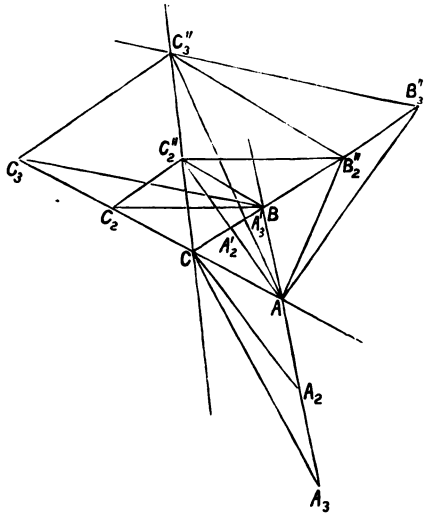
$$K/(1+K) = x,$$

the range of which is from 0 to 1; then, taking $m = 1/x$, we have $m > 1$. Produce BA, AC, CB indefinitely, and mark off corresponding multiples of them, such as

$$BA_2 = 2BA,$$

$$BA_3 = 3BA, \ldots.$$

Then CA_2 will be parallel to AA'_2, CA_3 to AA'_3, ..., each being m times as long. BC_2, BC_3, ... and AB_2, AB_3, ... follow the same law. Let AB_3 remain as AB''_3, but draw $C''_3B''_3$ parallel and equal to BC_3. Then AC''_3 will be parallel and equal to CA_3; for $C_3C''_3$ is parallel and equal to BB''_3 and to CB''_2;



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therefore $B_2''C_2''$ is parallel and equal to CC_2 and to C_2A, and, if C, C_2'' be joined and BC_2'' be drawn parallel to $B_2''C_2''$, we have

$$BC_2'' = \tfrac{1}{2}B_2''C_2'' = CA;$$

therefore CC_2'' is parallel and equal to AA_2, and $AB_3''C_3''$ is nine times the triangle which has AA_2', BB_2', CC_2' for its sides. Similar reasoning applies to any value of m. We further have the following law :—

$$\Delta ACB_m'' = m\Delta ABC,$$

and $$\Delta AB_m''C_m'' = m\Delta AA_m'B_m'' = m\,[m-(m-1)/m]\,\Delta ABC.$$

Therefore the triangle A″B″C″ of the Question is

$$[(m^2-m+1)/m^2]\,\Delta ABC = (1-x+x^2)\,\Delta ABC.$$

Now, the differential coefficient of $1-x+x^2$ is $2x-1$, which, taken as zero, makes $x = \frac{1}{2}$ or $K = 1$.

15524. (R. Chartres.) (Suggested by Question 15447).—Find the mean value of the angle subtended by BC at all points of the area BAC, (1) when BAC is a segment of a circle and $A = \pi/n$, (2) when BAC is any triangle.

Solution by the Proposer.

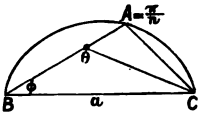
(1) Taking the radius of the circle = 1,

$$M(\theta) = \pi - 2M(\phi)$$

$$= \pi - 2\int_0^{\pi-A} \phi \cdot \tfrac{1}{2}\,\frac{a^2\sin^2(A+\phi)}{\sin^2 A}\,d\phi/\text{area}$$

$$= \pi - 2\int \phi\,[1-\cos(2A+2\phi)]\,d\phi/\text{area}$$

$$= \pi - 2\,\frac{[\pi^2(1-1/n)^2 - \sin^2\pi/n]}{2\pi(1-1/n)+\sin 2\pi/n}.$$

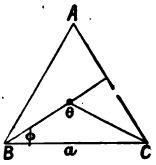


If $n = 2$, then $M(\theta) = \frac{1}{2}\pi + 2/\pi$.

(2) $$M(\phi) = \int_0^B \phi \cdot \tfrac{1}{2}r^2 d\phi/\Delta$$

$$= \frac{a\sin C}{b}\int_0^B \phi\,\operatorname{cosec}^2(\phi+C)\,d\phi$$

$$= \frac{a\sin C}{b}\,(B\cot A + \log a/c),$$



with a similar expression for the other base angle;

therefore $$M(\theta) = \pi - \left\{\frac{\cos A}{bc}(c^2B + b^2C) + \frac{\sin A}{bc}(c^2\log a/c + b^2\log a/b)\right\}.$$

(i.) If the triangle is right-angled isosceles,

$$M(\text{supplement of } \theta) = \log 2.$$

(ii.) If the triangle is equilateral, $M(\theta) = \frac{1}{3}\pi$, which is evident since the sides subtend equal angles whose sum is 2π.

(iii.) Hence, also, if the figure is a regular polygon of n sides, the $M(\theta)$ ought to be $2\pi/n$.

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15127. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorise completely into *prime* factors $N = 10^{24} + \frac{1}{4}7\,10^{12} + 1$.

Solution by the PROPOSER *and* JAMES BLAIKIE, M.A.

Let $N = x^8 + \frac{1}{4}7x^4y^4 + y^8$; therefore $4N = (x^4 + 4y^4)(y^4 + 4x^4)$. Each factor is a Bin-Aurifeuillian, the resolution of which is known to be

$$4N = \{(x-y)^2 + y^2\}\{(x+y)^2 + y^2\}\{(y-x)^2 + x^2\}\{(y+x)^2 + x^2\}.$$

Now take $x = 1000$, $y = 1$; then N becomes the given number, and

$$4N = (999^2 + 1^2)(1001^2 + 1^2)(999^2 + 1000^2)(1001^2 + 1000^2)$$
$$= 998002 \,.\, 1002002 \,.\, 1998001 \,.\, 2002001;$$

therefore $N = (23 \,.\, 21737)(501001)(277 \,.\, 7213)(2002001)$, all primes.

Note.—Similarly may be factorised

$$N = 3^{48} + 68 \,.\, 3^{24} + 2^8; \quad \text{taking } x = 3^6,\ y = 2.$$
$$N = 5^{32} + 68 \,.\, 5^{16} + 2^8; \quad \text{taking } x = 5^4,\ y = 2.$$

15381. (Rev. T. WIGGINS, B.A.)—At what times between the hours x and $x+1$ and between y and $y+1$ are the minute and hour hands in the former case in the same position as the hour and minute hands respectively in the latter case?

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E., *and* A. M. NESBITT, M.A.; (II.) *by* H. J. WOODALL, A.R.C.Sc.

(I.) Let x hr. $+ \xi$ min., y hr. $+ \eta$ min. be the times indicated. Let α, β be the angular distances of minute hand from 0 hr. at those times; a, b be the angular distances of hour hand from 0 hr. at those times; a', b' be the angular distances of hour hand from x hr., y hr. at those times (all angles measured in degrees). Here x, y always < 12, and ξ, η always < 60; so that 0 hr. means 12 o'clock.

Now, 1 minute space of minute hand $= \frac{1}{60}(360°) = 6°$, 1 hour space of hour hand $= \frac{1}{12}(360°) = 30°$; therefore

$$\alpha = 6\xi, \qquad \beta = 6\eta, \qquad a' = \tfrac{1}{12}\alpha = \tfrac{1}{2}\xi, \qquad b' = \tfrac{1}{12}\beta = \tfrac{1}{2}\eta,$$
$$a = 30x + a' = 30x + \tfrac{1}{2}\xi, \qquad b = 30y + b' = 30y + \tfrac{1}{2}\eta.$$

But $\alpha = b$, and $\beta = a$ (since the hands interchange positions); therefore $6\xi = 30y + \frac{1}{2}\eta$, and $6\eta = 30x + \frac{1}{2}\xi$. Solving these gives ($\xi$, η in *minutes*) $\xi = \frac{60}{143}(x + 12y)$, $\eta = \frac{60}{143}(y + 12x)$.

(II.) Let n be the number of minutes the hands are apart. While the hour hand moves over $60 - n$ minute spaces the minute hand moves over $60(y-x) + n$ minute spaces, and therefore gains $60(y-x-1) + 2n$ in this time. On ordinary clock faces the minute hand gains 11 minutes in 12; therefore

$$12\{60(y-x-1) + 2n\} = 11\{60(y-x) + n\}; \quad \therefore\ 60(y-x) + 13n = 720,$$

for example, shortly after 3 o'clock for the first position, about 12.15 for the second. Therefore $y - x = 9$; therefore $13n = 180$; therefore

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$n = 13\frac{1}{11}$. Now at 3 o'clock there is 15 minutes difference. The minute hand starting from the vertical position has to gain $1\frac{3}{12}$ minute spaces, and this it does in $\frac{11}{12} \times \frac{11}{3} = 1\frac{37}{143}$ min. Therefore the first position of the minute hand is 3 hr. $1\frac{37}{143}$ min., of the hour hand 3 hr. $15\frac{15}{143}$ min.; the second position of the minute hand is 12 hr. $15\frac{15}{143}$ min., of the hour hand 12 hr. $1\frac{37}{143}$ min.

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*Primitive Roots applied to the Pellian Equation.**

By R. W. D. Christie.

The objects of the paper are (1) to ease the work of solution, (2) to show the connexion with the primitive roots.

I. ($p = 4m+1$).—Let $AB-CD = 1$; then, if $C = A+B$, we have

$$A(C-A)-C(A-D_1) = 1 \quad \text{or} \quad A^2+1 = CD_1.$$

the type of the Pellian.

Now, by Euler's theorem of the two squares adapted, we have

$$(ac+bd)[a^2+b^2-ac-bd]-(a^2+b^2)[ac+bd-c^2-d^2] = 1 \quad \text{.....(A)},$$

when $ad-bc = \pm 1$; or, more generally still, supposing p composite

$$= (a^2+b^2)(c^2+d^2),$$

$$(ac+bd)[(a^2+b^2)(c^2+d^2)-(ac+bd)]-[(a^2+b^2)(c^2+d^2)][ac+bd-1] = 1.$$

Now, from the type, if $A = ac+bd$, $C = a^2+b^2$, we have

$$\pm(ac+bd)^2+1 = (a^2+b^2)(c^2+d^2) \quad \text{.....(a), (b)},$$

when $$ad-bc = \pm 1.$$

From each of these we can form an endless chain of integral equations

$$(a^2+b^2+ac+bd)^2+1 = (a^2+b^2)[a^2+b^2+c^2+d^2+2ac+2bd] \text{.....(a)},$$

$$(a^2+b^2-ac-bd)^2+1 = (a^2+b^2)[a^2+b^2+c^2+d^2-2ac-2bd] \text{.....(b)},$$

$$[2(a^2+b^2)+ac+bd]^2+1 = (a^2+b^2)[4(a^2+b^2)+4(ac+bd)+c^2+d^2] \text{...(a}_1\text{)},$$

$$[2(a^2+b^2)-ac-bd]^2+1 = (a^2+b^2)[4(a^2+b^2)-4(ac+bd)+c^2+d^2] \text{...(b}_1\text{)},$$

and generally

$$[n(a^2+b^2)\pm(ac+bd)]^2+1 = (a^2+b^2)[n^2(a^2+b^2)\pm 2n(ac+bd)+c^2+d^2] \text{...(B)}.$$

This is the general form of the Pellian with arbitrary letters corresponding to $X^2+1 = PY^2$. Introducing the roots, the corresponding forms are

$$r^m \cdot r^{3m} - p[(p^2-r^{4m}-r^{2m}-2)/2p]+1 \quad \text{.....(A)},$$

a form adapted to the residues

$$(r^m)^2+1 = p(c^2+d^2), \quad (r^{3m})^2+1 = p(c_1^2+d_1^2) \quad \text{.....(a, b)},$$

$$(pq\pm r^m)^2+1 = p[pq^2\pm 2qr^m+c^2+d^2] \quad \text{.....(B)},$$

where q is arbitrary, the general form of the Pellian in its root form.

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* Abstract of paper read before the Liverpool Mathematical Society, December 9, 1903.

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E.g., let $p = 73$; then $a = 8$, $b = 3$; then $8d - 3c = 1$ gives $c = 5$, $d = 2$ and $8d_1 - 3c_1 = -1$ gives $c_1 = 3$, $d_1 = 1$; consequently

$$46 \cdot 27 - 73 \cdot 17 = 1 \quad \text{.....(A)},$$

$$46^2 + 1 = 73 \cdot 29, \quad 27^2 + 1 = 73 \cdot 10 \quad \text{.....(a, b)},$$

$$(73q \pm 46)^2 + 1 = 73[73q^2 \pm 2q \cdot 46 + 29] \quad \text{.....(B)}.$$

Taking $q = 14$, we have

also
$$\left.\begin{aligned} 1068^2 + 1 &= 73 \cdot 125^2, \quad \text{the quadratic form} \\ 976^2 + 1 &= 73 \cdot 13049, \quad \text{the non-quadratic form} \end{aligned}\right\} r^m \text{ values},$$

$$\left.\begin{aligned} 1049^2 + 1 &= 73 \cdot 15074 \\ 995^2 + 1 &= 73 \cdot 13562 \end{aligned}\right\} r^{3m} \text{ values}.$$

To exemplify the principles of solution I shall take an example of moderate difficulty, viz., $X^2 - 73Y^2 = 1$. The convergents are $\frac{8}{1}$, $\frac{9}{1}$, $\frac{17}{2}$, $\frac{94}{11}$, no more being necessary. Now

$X_1 = 17 \cdot 2 + 94 \cdot 11 = 1068$, $Y_1 = 2^2 + 11^2 = 125$ [*v. infra*, III. (3)];

consequently $(2 \cdot 1068^2 + 1)^2 - 73(2 \cdot 125 \cdot 1068)^2 = 1$,

or $2281249^2 - 73 \cdot 267000^2 = 1$,

the least integral solution.

II. Here I have assumed several theorems established as follows:—

(1) $r_n q_{n+1} - q_n r_{n+1} = \pm 1$ (r_n being remainder of p_n/q_n).

It is known that $p_n q_{n+1} - q_n p_{n+1} = \pm 1^{n-1}$. Let

$p_{n+1} = q_{n+1} m_{n+1} + r_{n+1}$ (m_n being quotient), and $p_n = q_n m_n + r_n$;

then $p_{n+1} q_n = q_n q_{n+1} m_{n+1} + q_n r_{n+1}$ and $p_n q_{n+1} = q_n q_{n+1} m_n + r_n q_{n+1}$;

then $p_n q_{n+1} - p_{n+1} q_n = q_n q_{n+1}(m_n - m_{n+1}) - r_n q_{n+1} - q_n r_{n+1} = \pm 1$.

But $m_n = m_{n+1} =$ integer of $\sqrt{p}$; therefore $r_n q_{n+1} - q_n r_{n+1} = \pm 1$.

(2) In the equation $X^2 - PY^2 = -1$, $Y = q_{n+1}^2 + q_n^2$. We have

$$p_n^2 - Pq_n^2 = +b_n, \quad p_{n+1}^2 - Pq_{n+1}^2 = -b_n;$$

then, by addition,

$$p_n^2 + p_{n+1}^2 = P(q_n^2 + q_{n+1}^2), \quad p_{2n+1}^2 + 1 = P(q_n^2 + q_{n+1}^2)^2;$$

therefore $$p_{2n+1}^2 + 1 = (p_{n+1}^2 + p_n^2)(q_{n+1}^2 + q_n^2);$$

thus, by Euler's theorem,

$$\begin{aligned} p_{2n+1}^2 + 1 &= (p_{n+1} q_{n+1} - p_n q_n)^2 + (p_{n+1} q_n + p_n q_{n+1})^2 \\ &= (p_{n+1} q_{n+1} + p_n q_n)^2 + (p_{n+1} q_n - p_n q_{n+1})^2. \end{aligned}$$

Hence $p_{2n+1} = X = p_{n+1} q_{n+1} + p_n q_n$, $Y = q_n^2 + q_{n+1}^2$.

(3) The law of the multipliers applies also to the remainders. We have $q_n = m q_{n-1} + q_{n-2}$ by the general theory. Let $r_n = m r_{n-1} + r_{n-2}$;

then $$q_n r_{n-1} = m q_{n-1} r_{n-1} + q_{n-2} r_{n-1};$$

also $$r_n q_{n-1} = m q_{n-1} r_{n-1} + q_{n-1} r_{n-2};$$

therefore $$q_n r_{n-1} - r_n q_{n-1} = q_{n-2} r_{n-1} - q_{n-1} r_{n-2} = \pm 1.$$

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III. Another mode of attacking the problem is the following:—Let $X^2-(a^2+b^2)(c^2+d^2)^2=-1$; thus

$$X^2+1 \equiv [a(c^2-d^2)+2bcd]^2+[2acd-b(c^2-d^2)]^2 \quad \ldots\ldots(1),$$
$$\equiv [a(c^2-d^2)-2bcd]^2+[2acd+b(c^2-d^2)]^2 \quad \ldots\ldots(2),$$
$$\equiv [c(ac+bd)+d(ad-bc)]^2+[c(ad-bc)-d(ac+bd)]^2 \quad \ldots(3),$$
$$\equiv [c(ac+bd)-d(ad-bc)]^2+[c(ad-bc)+d(ac+bd)]^2 \quad \ldots(4).$$

In the previous example we have $a=8$, $b=3$, $c=$ convergent 11, $d=2$; thus $1068^2+1=1068^2+1=804^2+703^2=1000^2+375^2=73.125^2$.

It will be noticed that the squares (3) contain Y, and that either $[2acd-b(c^2-d^2)]$ or its reverse $[a(c^2-d^2)-2bcd]$ must equal ± 1.

Exemplification for the equation $X^2-73Y^2=\pm 1$:

	0,	8,	1,	7,	8,	7,	1,	8;
	1,	9,	8,	3,	3,	8,	9,	1;
	8,	1,	1,	5,	5,	1,	1,	16;
$p_n=$	8,	9,	17,	94,	487,	581,	1068;	
$q_n=$	1,	1,	2,	11,	57,	68,	125.	

The corresponding forms of the convergents are:—

$$\frac{c(6b-a)\sim d(6a+b)}{6d-c},\quad \frac{c(5b-a)\sim d(5a+b)}{5d-c},\quad \frac{c(b)-d(a)}{d},\quad \frac{c(a)+d(b)}{c},$$
$$\frac{c(5a+b)+d(5b-a)}{5c+d},\quad \frac{c(6a+b)+d(6b-a)}{6c+d},\quad \frac{c(ac+bd)+d(bc-ad)}{c(c)+d(d)},$$

the type of the convergents to the sinister of d being

$$c(bx-ay)\sim d(ax+by),$$

and that to the dexter of c being $c(ax+by)+d(bx-ay)$, the reverse. Hence, it will be seen, the whole scheme of solution depends upon c and d.

IV. ($p=4m-1$).—For this form see my solution given, Vol. LXX., p. 51. Corresponding equations are

$$(rac+bd)[ra^2+b^2-(rac+bd)]-(ra^2+b^2)[rac+bd-(rc^2+d^2)]=r \ldots(A),$$

when $ad-bc=\pm 1$, *i.e.*, $r^{m+1}.r^{3m+2}-p.q=r$ (a root).....(B).

General Forms.

$$[2a^2n+2an+2a+n+1]^2+1=(2a^2+2a+1)[(na+1)^2+(na+n+1)^2] \ldots(1),$$

with corresponding root equations

$$[(n+1)r^m+r^{3m}]^2+1=p[pn^2+2nr^m+2] \quad (n=\text{anything}) \ldots\ldots(2),$$
$$[(n+1)r^{3m}+nr^m]^2+1=p[pn^2+(2n+1)r^{3m}-2a+1] \quad \ldots\ldots\ldots(3).$$

The special form

$$(4a^3+6a^2+6a+2)^2+1=(4a^2+4a+5)[(a+1)^2+a^2]^2 \quad (a \text{ arbitrary})$$

gives interesting convergents in the root form

$$\frac{p_n}{q_n}=\frac{r^{3m}-r^m}{1},\ \frac{\mp r^m\pm 1}{\frac{1}{2}(r^{3m}-r^m-1)},\ \frac{\mp r^{3m}\pm 1}{\frac{1}{2}(r^{3m}-r^m+1)},\ \frac{p-2}{\mp r^{3m}\pm r^m},\ \frac{pa\pm r^m}{\frac{1}{2}(p-3)}.$$

Consequently $(pa\pm r^m)^2+1=p[\frac{1}{2}(p-3)]^2$; let $a=7$; then $p=229$; thus $(229.7+107)^2+1=229.113^2$, and three other non-quadratic forms. Other useful forms for study are $p=2a^2+2a+1$.

$$[2na^2+2(n+1)a+n+1]^2+1=p[2n^2a^2+2n(n+2)a+(n+1)^2+1] \ldots(1),$$
$$[2na^2+2(n-1)a+n-1]^2+1=p[2n^2a^2+2n(n-2)a+(n-1)^2+1] \ldots(2),$$

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when n is anything,

$$(2k^2-k+2)(2k^2+k+2)-(4k^2+3)(k^2+1)=1 \quad (k \text{ arbitrary}).....(3),$$

$$[4k^4+4k^3+5k^2+2k+2][2k^2+k+2]-[(k^2+k+1)(4k^2+3)][2k^2+k+1]=1 \quad .....(4).$$

Variant Solution.

We have $2aed-b(c^2-d^2)=\pm 1$, where $p=a^2$ (even) $+b^2$ (odd) $=8^2+3^2$, say; then $16cd-3(c^2-d^2)=\pm 1$ or $16.1-3.5=1$ and $cd=3k+1$, $c^2-d^2=16k+5$ (odd). Thus $2c^2$ or $2d^2=16k+5\pm[(16k+5)^2+(6k+2)^2]^{\frac{1}{2}}$; here $k=y$, $c=11$, $d=2$; also the convergents

$$(ac+bd)/c,\ (bc-ad)/d,\ =\tfrac{9}{11}\tfrac{4}{7},\ \tfrac{1}{2}\tfrac{7}{5}.$$

Thus $X=c(ac+bd)+d(bc-ad)=11.94+2.17=1068$,

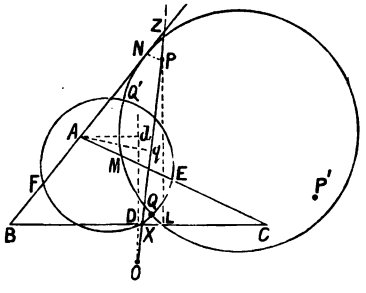
$Y=c^2+d^2=11^2+2^2=125$,

whence $1068^2+1=73.125^2$, the solution of $X^2+1=pY^2$.

14845. (C. E. Youngman, M.A.)—In a triangle ABC, if DEF be the pedal triangle of the circum-centre O, and LMN that of any other point P, show that there exists a point Q making the triangles QDL, QEM, QFN similar to AOP, BOP, COP respectively; and hence that the circles DEF and LMN will touch each other when the sum of the angles PAB, PBC, PCA is a right angle.

Solution by the Proposer.

Let d, q be the projections of A on OD, OP, and Q the image of q for EF. Then d, q are concyclic with AOEF, and therefore Q with DEF; and



$$\angle QEM = \angle C + \angle QEF$$
$$= \angle C + \angle qEF$$
$$= \angle BOF + \angle qOF$$
$$= \angle BOP;$$

also $QE : BO$
$= qE : AO = \sin qOE$
$= EM : OP;$

therefore the triangles QEM, BOP are similar. And in much the same way it may be shown that QFN, QDL are similar to COP, AOP.

We can now prove $\angle MQN = \angle MLN$; so that Q lies on the cirele LMN. Let DEF and LMN meet again at Q'; then

$$\angle QQ'L = \angle QNL = \angle BNL - \angle FNQ = \angle BPL - \angle OPC;$$

and $\angle QQ'D = \angle QFD = \angle qFd = \angle qOd = \angle OPL;$

therefore $\angle DQ'L = \angle BPL - \angle CPL = \angle PCB - \angle PBC.$

Thus for Q and Q' to coincide we must have

$$\angle DQL = \angle PCB - \angle PBC \quad \text{or} \quad \angle OAP = \angle C + \angle PCA - \angle PBC;$$

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which gives $\angle PAB + \angle PBC - \angle PCA = 90°$.

The negative sign for the angle PCA is accounted for by the position of P in the figure; make the convention that the sign of an angle PAB is to be the same as that of the triangle PAB, and the relation becomes symmetrical.

The properties here used furnish immediate proofs of several other theorems. For instance:

(1) If the vertices of a triangle be projected upon any line through its circum-centre, and thence upon the opposite sides, the last three projectors meet on the nine-point circle (*Mathesis*, 1896, p. 57).

(2) The four nine-point circles of PABC, and the four pedal circles of P in ABC, A in PBC, ... all meet at Q′ (Casey's *Sequel*, 6th ed., p. 251). For $\angle DQ'L = \angle PCB - \angle PBC$ is the condition that Q′ should lie on the nine-point circle of PBC, and the rest follows by symmetry.

(3) Let OP cut the sides at X, Y, Z. Circles on AX, BY, CZ as diameters meet at Q. For X is the double point of QDL and AOP (because it divides DL and OP both in the same ratio); therefore $\angle XQA = \angle XDO = 90°$, and so on. [The orthocentre lies on QQ_1, the common chord of these circles; hence Q_1 lies on the circum-circle, and another way of putting the proposition is: Perpendiculars from Q_1 to AQ_1, BQ_1, CQ_1 cut BC, CA, AB upon a straight line through O. See *Reprint*, Vol. XXVI., p. 92.]

(4) The pedal circles of all points on the same line OP pass through Q. For the position of Q depends only on that of q.

(5) When a point P and its isogonal P′ lie on the same diameter of ABC their common pedal circle touches the nine-point circle (M'Cay, *Trans. Royal Irish Academy*, 1889). For always the position of Q depends on the direction of OP, and that of Q′ on the direction of OP′.

15330. (A. M. Nesbitt, M.A.)—If (ξ, η, ζ) be the co-ordinates of any point on a tangent to the curve

$$(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = ax^2 + by^2 + cz^2 = a + b + c,$$

prove that $(m\zeta - n\eta)^2 + (n\xi - l\zeta)^2 + (l\eta - m\xi)^2 = 2$, l, m, n being the direction-cosines of the tangent. [N.B.—The *abc* in the Question as originally proposed is erroneous.]

Solution by the Proposer.

We have at once $\Sigma x^2 = 2$ from the given equations to the curve. The equations to any tangent line are $(\xi - x)/l = (\eta - y)/m = (\zeta - z)/n$, l, m, n (got by differentiating the equations to the curve) being given by

$$(b+c)lx + (c+a)my + (a+b)nz = 0, \quad alx + bmy + cnz = 0,$$

which at once yield $lx + my + nz = 0$. But

$$m\zeta - n\eta = mz - ny, \quad l\eta - m\xi = ly - mx, \quad n\xi - l\zeta = nx - lz;$$

whence $\Sigma(m\zeta - n\eta)^2 = \Sigma(mz - ny)^2 = x^2(m^2 + n^2) + \ldots + \ldots - 2\Sigma mnyz$

$$= \Sigma r^2 - (lx + my + nz)^2 = 2.$$

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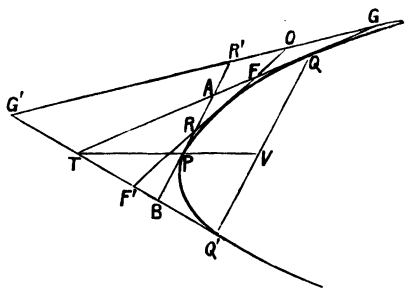
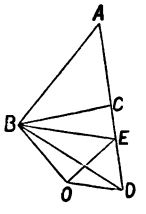
15456. (D. Biddle.)—Three straight lines lie on a plane, no two of them parallel. Through a given point external to them draw (1) a straight line of which the intercept between two of the given lines shall be bisected by the third; (2) two straight lines forming with two of the given lines a quadrilateral, which is bisected by the third.

Solution by A. M. Nesbitt, M.A.



The first part of Mr. Biddle's Question may be readily solved by using the fact that, if a tangent to a parabola meets two others which issue from a point T, in F and F', then FF' is bisected by the tangent at the point where the diameter through T meets the curve. Thus, if TA, TB, AB be the three given straight lines, and O the given point, we have merely (1) to find the focus and directrix of the parabola touching TA, TB, and AB, the last at its middle point P, *i.e.*, of the parabola through QQ'V (TQ = 2TA and TQ' = 2TB) and having its axis parallel to TP; (2) to draw the pair of tangents to this parabola from O. Of these (1) is solved in Besant's *Elementary Conic Sections*, Art. 213, and the other in the same work, Art. 34. If OFF' and GOG' be the two tangents meeting TA in F, G, and TB in F', G', and AB in RR', then FR = RF', GR' = R'G'.



The second part of the Question admits of many solutions; perhaps the following is the simplest. Let ABC be the triangle formed by the given lines, O the given point. Produce AC to D, making

$$CD = CA.$$

Join OD and draw BE parallel to OD. Then the triangles OEB, DEB are equal; therefore

$$\Delta ABC = \Delta DBC = OECB.$$

Thus OB and OE have been drawn as desired, and BC bisects the quadrilateral OBAE.

15366. (A. M. Nesbitt, M.A.)—The tangents at P, Q, R to the ellipse $x^2/a^2 + y^2/b^2 = 1$ form a triangle P'Q'R', which is inscribable in $x^2/a^2 + y^2/\beta^2 = 1$. Prove that PP', QQ', RR' meet in a point whose locus is $a^2b^2(b+\beta)^2x^2 + a^2\beta^2(a+\alpha)^2y^2 = a^2b^2(a\beta - \alpha b)^2$.

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Solution by Professor SANJÁNA, M.A.

Let the eccentric angles of P, Q, R on the first ellipse be α_1, β_1, γ_1 respectively. Then, for P′,

$$\frac{x}{a} = \frac{\cos\frac{1}{2}(\beta_1+\gamma_1)}{\cos\frac{1}{2}(\beta_1-\gamma_1)}, \qquad \frac{y}{b} = \frac{\sin\frac{1}{2}(\beta_1+\gamma_1)}{\cos\frac{1}{2}(\beta_1-\gamma_1)},$$

with similar co-ordinates for Q′ and R′. But these points are on the second ellipse; hence

$$a^2\beta^2\cos^2\tfrac{1}{2}(\alpha_1+\beta_1) + b^2\alpha^2\sin^2\tfrac{1}{2}(\alpha_1+\beta_1) = \alpha^2\beta^2\cos^2\tfrac{1}{2}(\alpha_1-\beta_1),$$

and two similar equations. By rearrangement we get the equation

$$\cos\alpha_1\cos\beta_1(\alpha^2\beta^2-a^2\beta^2+b^2\alpha^2) + \sin\alpha_1\sin\beta_1(\alpha^2\beta^2+a^2\beta^2-b^2\alpha^2) + a^2\beta^2-a^2\beta^2-b^2\alpha^2 = 0$$

and two others. Hence, by cross multiplication,

$$\frac{\alpha^2\beta^2-a^2\beta^2+b^2\alpha^2}{\sin\alpha_1\cos\frac{1}{2}(\beta_1+\gamma_1)} = \frac{\alpha^2\beta^2+a^2\beta^2-b^2\alpha^2}{\cos\alpha_1\sin\frac{1}{2}(\beta_1+\gamma_1)} = \frac{\alpha^2\beta^2-a^2\beta^2-b^2\alpha^2}{-\sin\alpha_1\cos\alpha_1\cos\frac{1}{2}(\beta_1-\gamma_1)}; \quad \ldots.$$

Hence, at P′,

$$\frac{x}{a} = -\cos\alpha_1\frac{\alpha^2\beta^2-a^2\beta^2+b^2\alpha^2}{\alpha^2\beta^2-a^2\beta^2-b^2\alpha^2}, \qquad \frac{y}{b} = -\sin\alpha_1\frac{\alpha^2\beta^2+a^2\beta^2-b^2\alpha^2}{\alpha^2\beta^2-a^2\beta^2-b^2\alpha^2};$$

but P′ is on $x^2/a^2+y^2/\beta^2=1$, so that

$$\frac{\alpha}{a} = \pm\frac{\alpha^2\beta^2-a^2\beta^2+b^2\alpha^2}{\alpha^2\beta^2-a^2\beta^2-b^2\alpha^2}, \qquad \frac{\beta}{b} = \pm\frac{\alpha^2\beta^2+\ldots}{\alpha^2\beta^2-\ldots}.$$

From these we can readily prove $\alpha/a \pm b/\beta = \pm 1$, which can also be obtained from the invariants of the conics. Taking the positive signs in the present case (as both curves are ellipses), we have

$$\frac{\alpha}{a} = \frac{-\cos\alpha_1\cos\frac{1}{2}(\beta_1-\gamma_1)}{\cos\frac{1}{2}(\beta_1+\gamma_1)} = \ldots = \frac{\cos\beta_1\cos\frac{1}{2}(\gamma_1-\alpha_1)}{\cos\frac{1}{2}(\gamma_1+\alpha_1)} = -\frac{\cos\gamma_1\cos\frac{1}{2}(\alpha_1-\beta_1)}{\cos\frac{1}{2}(\alpha_1+\beta_1)} \quad \ldots\ldots \text{(i.)},$$

$$\frac{b}{\beta} = -\frac{\sin\alpha_1\cos\frac{1}{2}(\beta_1-\gamma_1)}{\sin\frac{1}{2}(\beta_1+\gamma_1)} = \ldots = \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)}.$$

The equation of PP′, joining $(a\cos\alpha_1,\ b\sin\alpha_1)$ to $(-\alpha\cos\alpha_1,\ -\beta\sin\beta_1)$ is

$$x(\beta+b)\sin\alpha_1 - y(\alpha+a)\cos\alpha_1 + \sin\alpha_1\cos\alpha_1(ab-a\beta) = 0;$$

of QQ′, $x(\beta+b)\sin\beta_1 - y(\alpha+a)\cos\beta_1 + \sin\beta_1\cos\beta_1(ab-a\beta) = 0$;

so for RR′. All three meet in a point by relations (i.) and (ii.). At this point

$$\frac{x(\beta+b)}{ab-a\beta} = -\frac{\cos\alpha_1\cos\beta_1\cos\frac{1}{2}(\alpha_1+\beta_1)}{\cos\frac{1}{2}(\alpha_1-\beta_1)} = \frac{\alpha}{a}\cos\alpha_1\cos\beta_1\cos\gamma_1,$$

$$\frac{y(\alpha+a)}{ab-a\beta} = \frac{\beta}{b}\sin\alpha_1\sin\beta_1\sin\gamma_1.$$

Hence $$\frac{\alpha^2x^2(\beta+b)^2}{a^2(ab-a\beta)^2} + \frac{\beta^2y^2(\alpha+a)^2}{b^2(ab-a\beta)^2} = \frac{\alpha^4}{a^4}\Pi\cos^2\alpha_1 + \frac{\beta^4}{b^4}\Pi\sin^2\alpha_1;$$

by relations (i.) and (ii.), and $\sin(\alpha_1+\beta_1)+\sin(\beta_1+\gamma_1)+\sin(\gamma_1+\alpha_1)=0$, which can be derived from them, we can prove the right side to be unity. Hence the proposition.

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15490. (R. Tucker, M.A.)—Find the envelope of a parabola which cuts a given parabola in four points three of which are given co-normal points.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by the* Proposer.

(I.) Let the given parabola be $y^2 = 4ax$, the other

$$(x+py)^2 + 2gx + 2fy + c = 0.$$

The ordinates of the four intersections are given by

$$y^4 + 8apy^3 + (16a^2p^2 + 8ag)\,y^2 + 32a^2fy + 16a^2c = 0.$$

If m be the tangent of the angle made by the normal at a point with the axis, the ordinate of the point is $-2am$; hence, if m_1, m_2, m_3 be the co-normals and m the fourth point, we have $\Sigma y_1 = -8ap$, so that $\Sigma m_1 + m = 4p$. Similarly, $\Sigma m_1m_2 + m\Sigma m_1 = 4p^2 + 2g/a$, $m\Sigma m_1m_2 + m_1m_2m_3 = 4f/a$, and $mm_1m_2m_3 = c/a^2$. But, if the normals at m_1, m_2, m_3 meet in (h, k), it is known that $\Sigma m_1 = 0$, $\Sigma m_1m_2 = 2 - h/a$, $m_1m_2m_3 = -k/a$; therefore $m = 4p$; also $c = -4pka$, $f = 2ap - ph - \frac{1}{4}k$, $g = a - \frac{1}{2}h - 2ap^2$. Thus the equation of the variable parabola is

$$(x+py)^2 + x\,(2a - h - 4ap^2) + y\,(4ap - 2ph - \tfrac{1}{2}k) - 4pak = 0$$

or $$p^2\,(y^2 - 4ax) + 2p\,(xy + 2ay - hy - 2ak) + x^2 + 2ax - hx - \tfrac{1}{2}ky = 0.$$

Hence the envelope is readily seen to be

$$(xy + 2ay - hy - 2ak)^2 = (y^2 - 4ax)(x^2 + 2ax - hx - \tfrac{1}{2}ky),$$

which is the cubic

$$xy^2\,(2a - h) + 4ax^3 + \tfrac{1}{2}ky^3 + 4ax^2\,(2a - h) - 6akxy + (2a - h)^2\,y^2 - 4aky\,(2a - h) + 4a^2k^2 = 0.$$

Note.—If we do not take the three co-normal points to be fixed, we get only one relation between the four parameters p, g, f, c, and the envelope can only be represented by its differential equation.

(II.) In my paper (*Proc. Lond. Math. Soc.*, Vol. xxi., Nos. 397, 398) on "Co-normal Points" the equation to the parabola is found to be

$$(y + \lambda x)^2 = \tfrac{1}{2}\,[ax\,(8 + \lambda^2 S_2)] + \tfrac{1}{2}\,[\lambda ay\,(2S_2 + \mu\lambda)] + 4\lambda\mu a^2,$$

where λ is arbitrary and $S_2 = \Sigma m^2$ and $\mu = m_1m_2m_3$. If the co-normal points are fixed, then the envelope is found from

$$\lambda^2\,[x^2 - (\tfrac{1}{2}S_2)\,ax - \tfrac{1}{2}\mu ay] + \lambda\,(2xy - S_2ay - 4\mu a^2) + y^2 - 4ax = 0.$$

Hence the curve is

$$2\,(2x^2 - S_2ax - \mu ay)(y^2 - 4ax) = (2xy - S_2ay - 4\mu a^2)^2,$$

a cubic curve (since the terms of the fourth degree vanish).

15141. (The late Rev. G. Richardson, M.A.)—If a be an edge of a tetrahedron, A the dihedral angle at that edge, Δ the area of a face, l the perpendicular distance between a pair of opposite edges, and V the volume, then $3V\Sigma(1/l^2) = \Sigma(a \cot A)$ and $9V^2\Sigma(1/l^3) = \Sigma(\Delta^2)$.

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Solution by R. F. DAVIS, M.A.

Let ABC be a triangle ; A, B, C its angles, and a, b, c its sides. Also let D be the vertex of a tetrahedron having ABC for its base ; DA $= p$, DB $= q$, DC $= r$. Then, if M, N be points in BC, DA respectively such that MN is a minimum and therefore perpendicular to both these opposite edges (BM $= x$, DN $= y$), from the triangle ABC,

$$c^2(a-x) + b^2x = a\left\{\mathrm{AM}^2 + x(a-x)\right\};$$

from the triangle DBC, $q^2(a-x) + r^2x = a\left\{\mathrm{DM}^2 + x(a-x)\right\}$;

whence $c^2(a-x) + b^2x - q^2(a-x) - r^2x = a\left\{(p-y)^2 - y^2\right\} = ap(p-2y)$,

$$(b^2 + q^2 - c^2 - r^2)\,x + 2apy = a(p^2 + q^2 - c^2).$$

Similarly, from the triangles BAD, CAD, we get

$$(b^2 + q^2 - c^2 - r^2)\,y + 2apx = p(a^2 + q^2 - r^2).$$

These equations are sufficient to determine x, y, MN^2. The work is rather lengthy ; but the final result is

$$\mathrm{MN}^2\left\{4a^2p^2 - (b^2 + q^2 - c^2 - r^2)^2\right\} = 144\mathrm{V}^2,$$

which easily leads to the second part of the Question.

If α, β, γ be the trilinear co-ordinates of E, the foot of the perpendicular from D on the plane of ABC,

$a\cot(\mathrm{BC}) + b\cot(\mathrm{CA}) + c\cot(\mathrm{AB}) = (a\alpha + b\beta + c\gamma)/\mathrm{DE} = 2\Delta/\mathrm{DE} = 2\Delta^2/3\mathrm{V}$.

There are three other similar relations, and $\Sigma a\cot(\mathrm{BC}) = \Sigma\Delta^2/3\mathrm{V}$.

15022. (Rev. Prebendary WHITWORTH, M.A.)—If n be a positive integer, show that

$$1 - \frac{2n}{1.3} + \frac{2^2n(n-1)}{1.3.5} - \frac{2^3n(n-1)(n-2)}{1.3.5.7} + \ldots = \frac{1}{2n+1},$$

and that

$$1 - \frac{n}{1.3} + \frac{n(n-1)}{1.3.5} - \frac{n(n-1)(n-2)}{1.3.5.7} + \ldots = \frac{1}{2n+1}\,\Sigma_0^n\left(\frac{\mathrm{C}_x^n}{\mathrm{C}_{2x}^{2n}}\right).$$

Solution by SOLIDUS.

The first part is proved directly by continued application of the identity

$$\frac{1}{2n+1} = \frac{1}{2r+1} - \frac{2(n-r)}{2r+1}\,\frac{1}{2n+1},$$

putting successively $r = 0, 1, 2, \ldots$. Thus

$$\begin{aligned}\frac{1}{2n+1} &= 1 - \frac{2n}{1}\,\frac{1}{2n+1} = 1 - \frac{2n}{1}\left(\tfrac{1}{3} - \frac{2(n-1)}{3}\,\frac{1}{2n+1}\right)\\ &= 1 - \frac{2n}{1.3} + \frac{2^2n(n-1)}{1.3}\left(\tfrac{1}{5} - \frac{2(n-2)}{5}\,\frac{1}{2n+1}\right)\\ &= 1 - \frac{2n}{1.3} + \frac{2^2n(n-1)}{1.3.5} - \frac{2^3n(n-1)(n-2)}{1.3.5.7}\,\frac{1}{2n+1},\end{aligned}$$

and so on.

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To prove the second part, note that the series $1-\frac{n}{3}+\frac{n(n-1)}{3.5}-\ldots$ is a particular case of the hypergeometric series, being $= F(1, -n, \frac{1}{2}, \frac{1}{2})$, while the series $\Sigma_0^n \frac{C_x^n}{C_{2x}^{2n}}$, which may be written

$$1+\frac{1}{2n-1}+\frac{1.3}{(2n-1)(2n-3)}+\ldots,$$

is $= F(1, \frac{1}{2}, \frac{1}{2}-n, -1)$. Now one of the standard identities in connexion with hypergeometric series may be written (see Forsyth, *Diff. Eq.*, 1st ed., § 127)

$$Y_1 = M_5 Y_5 + N_5 Y_6,$$

where $M_5 \equiv \frac{\Pi(\gamma-1)\,\Pi(\beta-\alpha-1)}{\Pi(\beta-1)\,\Pi(\gamma-\alpha-1)}$, $N_5 \equiv \frac{\Pi(\gamma-1)\,\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)\,\Pi(\gamma-\beta-1)}$,

$$Y_5 = y_{13} = (1-x)^{-\alpha} F[(\alpha, \gamma-\beta, \alpha-\beta+1, 1/(1-x)],$$

and $Y_1 = F(\alpha, \beta, \gamma, x)$. Now putting in the values $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}-n$, $x = -1$, these relations reduce to the following :—

$$M_5 = \frac{\Pi(-n-\frac{1}{2})\,\Pi(-\frac{3}{2})}{\Pi(-\frac{1}{2})\,\Pi(-n-\frac{3}{2})} = \frac{-n-\frac{1}{2}}{-\frac{1}{2}} = 2n+1,$$

$$N_5 = \frac{\Pi(-n-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\Pi(0)\,\Pi(-n-1)} = 0,$$

$$Y_5 = (1+1)^{-1} F(1, -n, \tfrac{3}{2}, \tfrac{1}{2}) = \tfrac{1}{2} F(1, -n, \tfrac{1}{2}, \tfrac{1}{2}),$$

and the identity $Y_1 = M_5 Y_5 + N_5 Y_6$ becomes

$$F(1, \tfrac{1}{2}, \tfrac{1}{2}-n, -1) = \tfrac{1}{2}(2n+1) F(1, -n, \tfrac{1}{2}, \tfrac{1}{2}),$$

which again may be written

$$\Sigma_0^n \frac{C_x^n}{C_{2x}^{2n}} = \frac{2n+1}{2}\left(1-\frac{n}{3}+\frac{n(n-1)}{3.5}-\ldots\right).$$

It would seem that the particular result deduced from Forsyth's formulæ is incorrect to the extent of a factor $\frac{1}{2}$. Will some reader indicate an error in the deduction, or, failing that, an error in the general formulæ given by Forsyth ?

It may be noted that the first part of the Question is a particular case of the well known formula

$$F(\alpha, \beta, \gamma, 1) = \frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)},$$

as is seen by putting $\alpha = 1$, $\beta = -n$, $\gamma = \frac{1}{2}$, $x = 1$.

The Proposer solves the second part as follows :—

Let a pack of cards consist of two suits of n cards each. Let x cards be drawn. The number of ways is C_x^{2n}. The number of ways consisting only of *pairs* is $C_{\frac{1}{2}x}^n$ if x be even, and zero if x be odd.

The number of ways in which an ace is *unpaired* is C_{x-1}^{2n-2}. The number of ways in which an ace and a deuce are unpaired is $2^2 C_{x-2}^{2n-4}$, and so on. Hence (*Choice and Chance*, Prop. xiv.) we have

$$C_x^{2n} - 2C_1^n C_{x-1}^{2n-2} + 2^2 C_2^n C_{x-2}^{2n-4} - \ldots = C_{\frac{1}{2}x}^n \ (x \text{ even}) = 0 \ (x \text{ odd}).$$

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Divide throughout by C_x^{2n}. Then give x every integral value from 0 to $2n$; add and divide throughout by $2n+1$, and in virtue of *Choice and Chance*, Excursus, Art. 10, we have the result required.

15513. (J. Macleod, H.M.I.S., I.S.O.)—Using the usual notation for the ellipse, if at S a perpendicular SD to the major axis is raised equal to SP and in this perpendicular a point E is taken such that SE = AN, prove that $\Delta TDE = \Delta ADS$. In what case does the tangent PT coincide with TD?

Solutions (I.) *by* G. G. Morrice, M.D., *and* J. Blaikie, M.A.; (II.) *by the* Proposer *and* A. M. Nesbitt, M.A.; (III.) *by* C. M. Ross *and* W. Scrimgeour, M.A., B.Sc.

(I.) $CX = a/e$, $CT = a^2/x$, $a - ex = e(CX - x)$,

$(a^2/x - ae)(a-x) = e(CX-x)(a^2/x-a)$, $(CT - CS)\,AN = CXN \cdot AT$,

$ST \cdot AN = SP \cdot AT$, $SP(ST - AS) = ST \cdot AN$, $ST(SP - AN) = AS \cdot SP$,

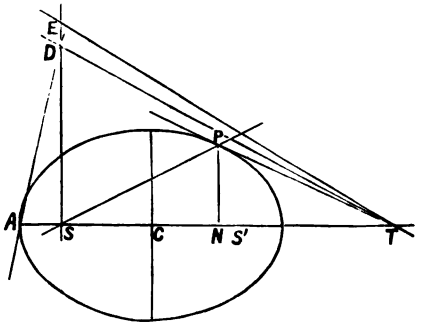
or $\frac{1}{2}ST(SD - SE) = \frac{1}{2}SD \cdot AS$;

therefore $\Delta TDE = \Delta ADS$. PT coincides with TD when P is an extremity of the latus rectum.

(II.) $AN : NA' = AT : TA'$,

$AN : AA' = AT : AA' + 2A'T$ .....(1),

$SP : ST = S'P : S'T$, $SP : S'P = ST : S''T$,



or $SP : AA' = ST : AA' + 2A'T$ .....(2).

From (1) and (2), $SP : AN = ST : AT$. Hence the theorem follows at once, and the tangent coincides with TD when N coincides with S.

(III.) 1. $SD = SP = a + e \cdot CN$, $SE = AN = a + CN$;

therefore $DE = CN(1 - e)$.

Again $ST = SC + CT = a(e + a/CN)$.

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Hence $\Delta TDE = \frac{1}{2}CN\,(1-e)\,a\,(eCN+a)/CN$
$= \frac{1}{2}a\,(1-e)(eCN+a) = \frac{1}{2}AS\,.\,SD = \Delta ADS.$

2. Again $\dfrac{SD}{ST} = \dfrac{a+eCN}{a\,(e+a/CN)} = \dfrac{CN}{a}$,

also $\dfrac{PN}{NT} = \dfrac{(SP^2-SN^2)^{\frac{1}{2}}}{CT-CN} = \dfrac{CN\,(1-e^2)^{\frac{1}{2}}}{(a^2-CN^2)^{\frac{1}{2}}}$ (after reduction).

Then, if PT and TD coincide, $CN/a = [CN\,(1-e^2)^{\frac{1}{2}}]/(a^2-CN^2)^{\frac{1}{2}}$; therefore $a^2-CN^2 = a^2-a^2e^2$; therefcre $CN = \pm ae = CS$ or CS'; *i.e.*, the tangent PT coincides with TD when the perpendicular PN passes through either focus.

15538. (W. H. Blythe, M.A.)—There are N bullets in a bag to be divided among n people. The first takes what he considers to be his share, finding one remainder, which he lays aside. The second, not knowing the first has removed any of the bullets, does exactly the same thing, and each of the people in turn repeats the process. Finally, the remaining bullets are divided into n equal portions, with one remainder, each person taking his share. Show that N is of the form An^p-n+1, where A is any positive integer and p is not less than $n+1$. Find an expression for the share of each person.

Solution by Lt.-Col. Allan Cunningham, R.E.

Let x_r = share taken by the r-th person (before final adjustment);
let N_r = number of bullets remaining after r-th operation (excluding those laid aside);
let x_r' = share received by r-th person after final adjustment;
let $\nu = (1-1/n)$, for shortness.
Then $x_1 = (N-1)/n$, $x_2 = (N_1-1)/n$, ..., $x_r = (N_{r-1}-1)/n$ ..... (1),
and $N_1 = N-x_1-1 = (N-1)-(N-1)/n = \nu\,(N-1)$.
Similarly, $N_2 = \nu\,(N_1-1) = \nu^2N-\nu^2-\nu$;
similarly, $N_3 = \nu\,(N_2-1) = \nu^3N-\nu^3-\nu^2-\nu$.
Lastly, $N_n = \nu^nN-(\nu^n+\nu^{n-1}+\ldots+\nu^2+\nu) = \nu^nN-\nu(\nu^n-1)/(\nu-1)$...(2).
Now let X = *additional* share given to each person on final division of the remaining (N_n+n) bullets (including the n previously laid aside).
Then $X = (N_n+n-1)/n$ ....(3).
By (2) and (3), $\nu^nN-\nu\,(\nu^n-1)/(\nu-1) = N_n = nX-(n-1)$.
But $\nu/(\nu-1) = -(n-1)$;
therefore $\nu^nN = nX-(n-1)-(n-1)(\nu^n-1) = nX-(n-1)\,\nu^n$;
therefore $N = nX/\nu^n-n+1 = n^{n+1}\,[X/(n-1)^n]-n+1$
$= n^{n+1}A-n+1$; [writing $A = X/(n-1)^n$], as required.
Also $x_r' = x_r+X = (N_{r-1}-1)/n+(N_n+n-1)/n$,
which, after some reduction, becomes $x_r' = (\nu^{r-1}+\nu^n)\,n^nA-1$.

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15428. (C. M. Ross.) (Suggested by Question 15376.)—Evaluate (1) $\int \cot x \cot 2x \cot 3x\, dx$, (2) $\int \tan x \tan 2x \tan 3x \tan 4x\, dx$; and generalize (2).

Solution by GEORGE SCOTT, M.A., *and the* PROPOSER.

(1) To find $\int \cot x \cot 2x \cot 3x\, dx = F(x)$. Let the product of cotangents be called $F'(x)$. $\cot 3x = \cot(2x+x) = (\cot 2x \cot x - 1)/(\cot 2x + \cot x)$; hence $\cot 2x \cot 3x = \cot 2x \cot x - \cot 3x \cot x - 1$; therefore

$$F'(x) = \cot^2 x (\cot 2x - \cot 3x) - \cot x = \cot^2 x \frac{\sin x}{\sin 2x \sin 3x} - \cot x,$$

$$\sin 2x \sin 3x = 2 \sin^2 x \cos x (3 - 4 \sin^2 x);$$

therefore
$$F'(x) = \frac{\cot x}{2 \sin^2 x (3 - 4 \sin^2 x)} - \cot x$$

or
$$= \tfrac{1}{3} \left\{ \frac{\cot x (1 + \cot^2 x)^2}{3 \cot^2 x - 1} \right\} - \cot x;$$

therefore
$$F(x) = -\tfrac{1}{3} \int \frac{(\cot x + \cot^3 x)\, d(\cot x)}{3 \cot^2 x - 1} - \int \cot x\, dx,$$

whence, by division,

$$F(x) = -\tfrac{1}{3} \int \cot x\, d(\cot x) - \tfrac{2}{3} \int \frac{\cot x\, d(\cot x)}{3 \cot^2 x - 1} - \int \cot x\, dx;$$

therefore $\int \cot x \cot 2x \cot 3x\, dx = -\tfrac{1}{3} \cot^2 x - \tfrac{1}{3} \log(3 \cot^2 x - 1) - \log \sin x$.

(2) To find $\int \tan x \tan 2x \tan 3x \tan 4x\, dx$. Let the integral be called $f(x)$ and its first derived $f'(x)$. To simplify $f'(x)$, since

$$\tan 4x = \tan(3x + x),$$

we have
$$\tan x \tan 3x \tan 4x = \tan 4x - \tan x - \tan 3x;$$

therefore
$$f'(x) = (\tan 4x - \tan x) \tan 2x - \tan 2x \tan 3x.$$

Now
$$(\tan 4x - \tan x) \tan 2x = \frac{2 \tan^2 2x}{1 - \tan^2 2x} - \frac{2 \tan^2 x}{1 - \tan^2 x}$$

or
$$= -\frac{2 \sin^2 2x}{\cos 4x} - \frac{2 \sin^2 x}{\cos 2x} = \frac{1 - \cos 4x}{\cos 4x} - \frac{1 - \cos 2x}{\cos 2x} = \frac{1}{\cos 4x} - \frac{1}{\cos 2x}.$$

Again, to simplify $\tan 2x \tan 3x$, since $\tan 3x = \tan(2x + x)$, we have

$$\tan 2x \tan 3x = (\tan 3x - \tan 2x - \tan x) \cot x$$

or
$$= \frac{3 - \tan^2 x}{1 - 3 \tan^2 x} - \frac{2}{1 - \tan^2 x} - 1$$

$$= \frac{4 - 1/\cos^2 x}{4 - 3/\cos^2 x} - \frac{2 \cos^2 x}{\cos 2x} - 1.$$

Hence
$$\tan 2x \tan 3x = \frac{2 \cos 2x + 1}{2 \cos 2x - 1} - \frac{1 + \cos 2x}{\cos 2x} - 1$$

$$= \frac{2}{2 \cos 2x - 1} - \frac{1}{\cos 2x} - 1,$$

after reduction. Thus

$$f(x) = \int \frac{dx}{\cos 4x} - \int \frac{dx}{\cos 2x} - 2 \int \frac{dx}{2 \cos 2x - 1} + \int \frac{dx}{\cos 2x} + x$$

or
$$= \int \frac{dx}{\cos 4x} - 2 \int \frac{dx}{2 \cos 2x - 1} + x,$$

$$\int \frac{dx}{2 \cos 2x - 1} = \int \frac{dx}{2(1 - 2 \sin^2 x) - 1} = \int \frac{dx/\cos^2 x}{1/(\cos^2 x) - 4 \tan^2 x} = \int \frac{d \tan x}{1 - 3 \tan^2 x},$$

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finally, $f(x) = \frac{1}{4}\log\tan(\frac{1}{4}\pi + 2x) - \frac{1}{\sqrt{3}}\log\left(\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right) + x,$

$$\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x} = \frac{\sin\frac{1}{6}\pi\cos x + \cos\frac{1}{6}\pi\sin x}{\sin\frac{1}{6}\pi\cos x - \cos\frac{1}{6}\pi\sin x} = \frac{\sin(\frac{1}{6}\pi + x)}{\sin(\frac{1}{6}\pi - x)};$$

therefore $\int \tan x \tan 2x \tan 3x \tan 4x\, dx$

$$= \tfrac{1}{4}\log\tan(\tfrac{1}{4}\pi + 2x) - \frac{1}{\sqrt{3}}\log\left\{\frac{\sin(\frac{1}{6}\pi + x)}{\sin(\frac{1}{6}\pi - x)}\right\} + x.$$

Note.—Each of the Solvers states that so far all efforts to generalize (2) have been unsuccessful.

15508. (R. F. Davis, M.A.) [Suggested by Mr. C. E. Youngman's Question 15435.]—Three circles touch each other externally and touch externally the circum-circle of a given triangle ABC at A, B, C respectively. Show that their radii are $R/(\cot B + \cot C - 1)$, ...

Solution by Frances E. Cave.

Let α, β, γ be the required radii. Then triangle of sides $R+\alpha$, $R+\beta$, $\alpha+\beta$ has angle 2C opposite to third side; therefore

$$\operatorname{cosec}^2 C = [(R+\alpha)(R+\beta)]/\alpha\beta,$$

with two similar equations; therefore

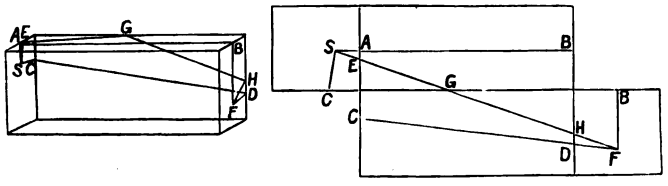
$$(R+\alpha)/\alpha = (\operatorname{cosec} C \operatorname{cosec} B)/\operatorname{cosec} A \text{ (since R and } \alpha \text{ are positive)}$$
$$= \cot B + \cot C;$$

therefore $\alpha = R/(\cot B + \cot C - 1).$

15532. (*Communicated by* R. F. Whitehead, B.A.)—The ends of a room are 12 ft. square and its length is 30 ft. Show that the shortest path a spider situated 3 ft. from the ceiling on the mid-line at one end of the room can take towards a fly at the diametrically opposite point on the other end wall is about 41 ft. 9½ in.

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by the* Proposer.

(I.) There are three paths by which the spider may reach the fly. He



may go by the ceiling (or floor) following the mid-line. In this case his

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path, SABF in the diagram, is clearly $3+30+9=42$ ft. Or he may go by either wall. In this case his path, SCDF in the diagram, is clearly $\sqrt{(42^2+6^2)} > 42$ ft. Lastly, he may go by both ceiling and wall, following the path SEGHF in the diagram, the distance in this case being $\sqrt{(39^2+15^2)} = 41{\cdot}785$ ft. $= 41$ ft. $9\frac{1}{2}$ in. nearly. It is interesting to observe that in this last case he has the choice of four minimum paths, as he may along with the ceiling choose either the right- or left-hand wall, or he may follow equal paths by either wall and the floor.

(II.) The spider may go to any one of four edges to begin with, and may end his journey from any one of four. When two such edges are chosen there is a single corresponding minimum path. The paths in which the spider goes to the floor and (1) ends from the ceiling or (2) ends from either side and those (3) and (4) in which he goes to either side and ends from either side must always be longer than (5) the floor or ceiling path. The remaining substantially distinct paths are those in which he goes to the ceiling and ends (6) from a side or (7) from the floor. If x be the length of the room, the squares of the lengths (5), (6), and (7) are

$$x^2+24x+144, \quad x^2+18x+306, \quad \text{and} \quad x^2+12x+612,$$
$$\alpha, \qquad \beta, \qquad \gamma.$$

If $x < 27$, $\alpha < \beta < \gamma$;
if $x > 27$ and < 51, $\alpha > \beta < \gamma$;
if $x > 51$, $\alpha > \beta > \gamma$.

Which of these three paths is the shortest depends thus on the length of the room.

15491. (L. Isserlis.)—On a circle centre O are taken points p, q, such that the arcs pq, qr are equal. Lines pP, qQ, rR are drawn parallel to a common tangent of the circle and a concentric conic meeting the latter in P, Q, R. Show that the tangent at Q to the conic is parallel to PR.

Solution by the Proposer *and* C. M. Ross.

Take the common tangent and the normal to the conic at the point of contact for axes, so that the equation of the conic may be written $ax^2+2hxy+by^2=2y$. Let θ be the angle which Op makes with the axis of y; then the co-ordinates of P are

$$x = 1/c^2(c\sin\theta + h\cos\theta - h), \quad y = a/c^2(1-\cos\theta),$$

where $$c^2 = ab-h^2.$$

If ϕ is the corresponding angle for r, the m of PR is

$$\frac{a(\cos\phi-\cos\theta)}{c(\sin\theta-\sin\phi)+h(\cos\theta-\cos\phi)} = \frac{a\sin\frac{1}{2}(\phi+\theta)}{c\cos\frac{1}{2}(\phi+\theta)-h\sin\frac{1}{2}(\phi+\theta)},$$

which is the same as the m of the tangent at the point Q, since the θ of Q is $\frac{1}{2}(\phi+\theta)$. Hence PR is parallel to the tangent at Q.

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15468. (R. Tucker, M.A.)—ABC is the circum-circle of a triangle. Ap, Ap' are lines drawn from A to p, p' on the circumference, each equal to the perpendicular from A on BC. Bq, Bq'; Cr, Cr' are like lines from B, C. Prove that pp', qq', rr' are the respective antiparallels to the angles A, B, C, and by their intersections form a triangle abc similar to the orthocentric triangle of ABC. Prove further that Aa, Bb, Cc intersect in K, the symmedian point of ABC.

Solution by Professor Sanjána, M.A.

Since $Ap = Ap'$, pp' is parallel to the tangent at A, *i.e.*, to the side EF of the pedal triangle; hence it is an antiparallel of the angle A. The lines qq', rr' are similarly parallel to FD, DE respectively; hence the triangle abc formed by the three is directly similar to the pedal triangle.

If the line through A perpendicular to pp' meet it in M, AM is a diameter of the circle; hence

$$AM = Ap^2/2R = 2R \sin^2 B \sin^2 C.$$

Also the distance of FE from A, $AN = 2R \cos A \sin B \sin C$. Thus NM, the distance between FE and $pp' = 2R \sin B \sin C \cos B \cos C$ = half the distance of D from FE. Hence b, c are the mid-points of DF, DE; so a is the mid-point of EF; and it is a well known theorem that Aa, Bb, Cc meet in the symmedian point of the given triangle.

15523. (A. M. Nesbitt, M.A.)—A man, who has m hats of his own in his hall, is visited by n friends, each wearing a hat. They leave their hats with those of their host. When they are going away they are unfortunately not in a condition to distinguish between one hat and another. Find the chance that no guest takes away his own hat.

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by* D. Biddle.

(I.) The first to leave finds $m+n$ hats in the hall. The chance that he takes his own is $1/(m+n)$, and the chance that he does not is $(m+n-1)/(m+n)$. The next to leave finds $m+n-1$ hats. The chance that his is one of them is clearly $(m+n-1)/(m+n)$. The chance that he takes his own is therefore $[1/(m+n-1)][(m+n-1)/(m+n)]$ or $1/(m+n)$, and the chance that he does not take his own is $(m+n-1)/(m+n)$. Similarly we may show that in the case of each guest the chance of not taking his own hat is the same, as is also evident *a priori*. The chance required is therefore $[(m+n-1)/(m+n)]^n$.

(II.) Let A, B, C, ... represent the n guests in the order of departure, that is, of taking hats. Then it is clear that A's chances of taking or leaving any particular hat are respectively $1/(m+n)$ and $(m+n-1)/(m+n)$. The latter is the chance that he leaves his own and takes some other.

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The chance that he takes B's hat is $1/(m+n)$, and in that case it becomes a matter of certainty ($=1$) that B goes without his own. But, if A leaves B's, of which the chance is $(m+n-1)/(m+n)$, then B's chance of taking some other than his own is $(m+n-2)/(m+n-1)$. Thus B's total chance is $(m+n-2)/(m+n)+1/(m+n)$, which is the same as A's. Similarly C's chance is $(m+n-3)/(m+n)+2/(m+n)$, because either A or B may have have taken C's hat, and so on. Consequently the required chance is $(m+n-1)^n/(m+n)^n$.

15492. (W. J. Greenstreet, M.A.)—Through the feet of the normals from a point P to a rectangular hyperbola passes another rectangular hyperbola such that its normals at these four points are concurrent. Find the point of concurrence and the locus of P if the two curves are equal.

Solution by A. M. Nesbitt, M.A.

The four normals from (h, k) to the equilateral hyperbola $xy-c^2=0$ meet the curve in $(cl, c/l)$, where l is a root of $cl^4-hl^3+kl-c=0$; and the equation to the line-pair (l_1l_2) and (l_3l_4) is

$$x^2-y^2+(l_1l_2+l_3l_4)(xy-c^2)-hx+ky=0.$$

Hence any conic through l_1, l_2, l_3, l_4 may be represented by

$$x^2-y^2+m(xy-c^2)-hx+ky=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the normal to (1) at the point (x, y) will pass through the point (θ, ϕ) if $(\theta-x)/(2x+my-h)=(\phi-y)/(mx-2y+k)$; so that the feet of the four normals from (θ, ϕ) to (1) lie on the conic

$$m(x^2-y^2)-4xy+x(k-m\theta+2\phi)+y(h+m\phi+2\theta)-h\phi-k\theta=0\equiv S.$$

If then they lie also on $xy=c^2$, it will be possible so to choose λ that

$$S+\lambda(xy-c^2)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

shall be the same conic as is represented by (1). Comparing coefficients of x^2, xy, and y^2, we see that $\lambda=m^2+4$, and with this substitution (2) becomes

$$m(x^2-y^2)+m^2xy-(m^2+4)c^2+x(k-m\theta+2\phi)+y(h+m\phi+2\theta)-h\phi-k\theta=0.$$

Thus we must have

$$k-m\theta+2\phi+mh=0, \quad h+m\phi+2\theta-mk=0, \quad 4c^2+h\phi+k\theta=0,$$

and these yield

$$\theta/(m^2h+3mk-2h)=\phi/(m^2k-3mh-2k)=1/(m^2+4),$$

where m is a root of the quadratic

$$2m^2(hk+2c^2)-3m(h^2-k^2)+4(4c^2-hk)=0.$$

There is no special difficulty in proving that, if the two hyperbolas are equal, the point (h, k) lies on the equilateral hyperbola

$$y^2-x^2+2xy=c^2\{4+(4+m^2)^{\frac{1}{2}}\},$$

m being, as before, a root of the above quadratic.

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15520. (Robt. W. D. Christie.)—Solve generally

$$A^3 + B^3 = (a^3 + b^3)(c^3 + d^3)$$

in integers.

Solution by Lt.-Col. Allan Cunningham, R.E.

A really general solution seems difficult; a solution of some generality may be obtained by taking $A = A'^2$, $a = a'^2$. Then

$$A'^6 + B^2 = (a'^6 + b^2)(c^2 + d^2) = (a'^3c \mp bd)^2 + (a'^3d \pm bc)^2;$$

whence $A'^3 = a'^3c \mp bd$, $B = a'^3d \pm bc$. Here A', a', c may be taken any integers; and b, d are found as any pair of co-factors of $(A'^3 \sim a'^3 . c)$; these now fix B.

The Proposer solves the Question as follows:—When r is a prime root of prime $4m-1$, $r^2 + r^{2m+1} = 0 \pmod p$, and there are other expressions; therefore

$$r^3 + (r^{m+1})^2 = 0 \pmod p \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Take any prime of that form, *e.g.*, $19 = 3^2 + 10$; thus

$$10^2(10 + 3^2) = 0 \pmod{19};$$

but, from (1), $3^3 + 7^2 = 0 \pmod{19}$; thus $10^3 + 30^2 = (3^2 + 4^2)(3^3 + 7^2)$. Similarly with other roots, and when one value is obtained of course others are got *ad libitum*; thus

$$(19 + 3)^3 + (19 + 7)^2 = 0 \pmod{19} \quad \text{and} \quad 22^3 + 26^2 = (10^2 + 7^2)(3^3 + 7^2).$$

Again $66^3 + 64^2 = (40^2 + 36^2)(3^3 + 7^2)$, ...

Other examples are $14^3 + 64^2 = (9^2 + 3^2)(3^3 + 7^2)$,

$14^3 + 26^3 = (4^2 + 2^3)(3^3 + 12^2)$, $(10^3 + 14^3)(5^2 + 2^3) = (7^3 + 18^3)(6^2 + 4^2)$.

15443. (Professor Neuberg.)—Trouver le lieu des centres des hyperboles dont les asymptotes sont parallèles à deux droites données u, v et qui divisent harmoniquement deux droites données AB, CD.

Solution by R. F. Whitehead, B.A., *and* A. M. Nesbitt, M.A.

Let the axes of reference be u, v; and let A, B, C, D be (x_r, y_r), where $r \equiv 1, 2, 3, 4$. The conic is of the form $2hxy + 2gx + 2fy + c = 0$. Since A, B and C, D are pairs of conjugate points,

$$h(x_1y_2 + x_2y_1) + g(x_1 + x_2) + f(y_1 + y_2) + c = 0, \ \ldots,$$

and, if the centre be (x, y),

$$hy + g = 0, \quad hx + f = 0.$$

Hence the locus is

$$x(y_1 + y_2 - y_3 - y_4) + y(x_1 + x_2 - x_3 - x_4) + x_3y_4 + x_4y_3 - x_1y_2 - x_2y_1 = 0.$$

15473. (C. E. Youngman, M.A.)—From A, B, C lines AEF′, BFD′, CDE′ are drawn making (all in the same sense) an angle θ with BC, CA, AB respectively; and, again, AE′F, BF′D, CD′E doing the same, but in

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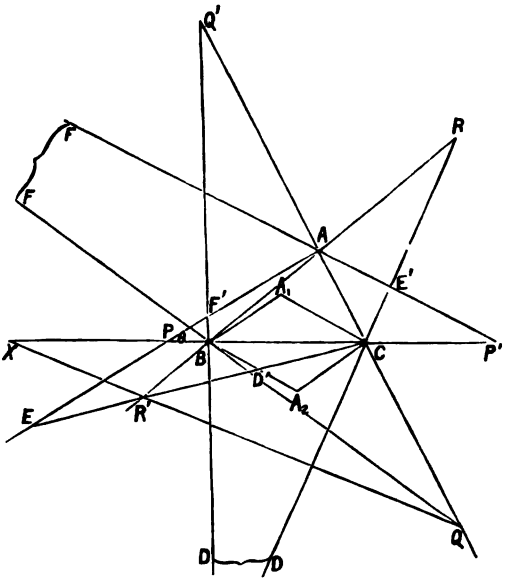
the other sense. Prove that ABC, DEF, D'E'F' are in perspective two by two, having the same axis, whose direction is constant, and that two of the centres P, P' lie on the Kiepert hyperbola and the third Q on the isogonal transformation of the Euler line OH. When $\theta = 60°$, Q is the "twin" of O.

Solution by Professor SANJÁNA, M.A.

The first part can be solved geometrically or analytically. Let the first set of lines meet BC, CA, AB in P, Q, R respectively, the second in



P', Q', R'; let QR' meet BC in X. Then, as $\angle BR'C = \angle BQC = \theta$, the quadrilateral BCQR' is cyclic; and D'AX, its diagonal triangle, is self-polar. Let tangents to this circle at B and C meet in A_1; then

$$\angle A_1BC = \angle BQC = \theta = \angle A'CB;$$

also A_1 is the pole of BC, and D' of AX, so that X is the pole of A_1D'. But it is the pole of AD'; hence A_1 lies on AD'. Thus we see that the line AD' is the join of A to the vertex of an isosceles triangle, of base angles each θ, described internally on BC. So for BE', CF'. But it is well known that these three joins meet on Kiepert's hyperbola in a point P whose co-ordinates are $\operatorname{cosec}(A-\theta)$, $\operatorname{cosec}(B-\theta)$, $\operatorname{cosec}(C-\theta)$. Similarly, BCRQ' is cyclic; the point where Q'R meets BC and the points A and D form its diagonal triangle; hence, if tangents to this circle at B and C meet in A_2, AD goes through A_2, where A_2BC is an isosceles triangle, of base angles each θ, described externally. Therefore the lines AD, BE, CF meet on Kiepert's hyperbola in a point P' whose co-ordinates

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are $\operatorname{cosec}(A+\theta)$, $\operatorname{cosec}(B+\theta)$, $\operatorname{cosec}(C+\theta)$. As these two sets of lines concur, D′E′F′ and DEF are in perspective with ABC.

To prove DEF and D′E′F′ in perspective, employ trilinear co-ordinates. It is found that D is the point $-a\sin(A-\theta) = \beta\sin(B+\theta) = \gamma\sin(C+\theta)$; and D′ is $-a\sin(A+\theta) = \beta\sin(B-\theta) = \gamma\sin(C-\theta)$; whence we get the equation of DD′ to be

$$a\sin(B-C)(\sin^2 A-\sin^2\theta)+\beta\sin B(\sin^2 B-\sin^2\theta) -\gamma\sin C(\sin^2 C-\sin^2\theta)=0.$$

The equations of EE′, FF′ are similarly known; and we see that the three lines meet in the point

$$a(\sin^2 A-\sin^2\theta)/\cos A = \beta(\sin^2 B-\sin^2\theta)/\cos B = \gamma(\sin^2 C-\sin^2\theta)/\cos C;$$

hence the result, from which we also see that the three triangles have the same axis of perspective.

The equation of this axis will be found to be

$$\Sigma a(\sin^2\theta-\sin^2 A)/\sin A = 0 \quad \text{or} \quad \Sigma\sin^2\theta\,.\,a/\sin A-\Sigma a\sin A=0;$$

thus this axis is always parallel to $\Sigma a/\sin A = 0$, that is, to the Lemoine line. It is seen that the three centres of perspective are collinear, the line of collinearity being

$$\Sigma a(\sin^2\theta-\sin^2 A)\sin(B-C)=0.$$

And the third centre Q lies on the hyperbola isogonally conjugate to Euler's line, viz.,

$$\beta\gamma\sin 2A\sin(B-C)+\gamma a\sin 2B\sin(C-A)+a\beta\sin 2C\sin(A-B)=0.$$

For $\Sigma\sin 2A\sin(B-C)/a$

$$\begin{aligned}&= (1/k)\,\Sigma\sin 2A\sin(B-C)(\sin^2 A-\sin^2\theta)/\cos A\\ &= (2/k)\,\Sigma\sin A\sin(B-C)(\sin^2\theta-\sin^2 A)\\ &= (2/k)\,\Sigma\sin A\sin(B-C)\sin^2\theta-2/k\,\Sigma\sin^3 A\sin(B-C),\\ &= 0.\end{aligned}$$

15359. (Robt. W. D. Christie.)—Without the use of the method of continued fractions, solve the equation $X^2-149Y^2=1$, and generalize your method.

Solution by the Proposer.

Using the method of primitive roots, we have

$$r^m\,.\,r^{3m}-p\left[\tfrac{1}{2}(p-s-t)\right]=1 \quad \text{.....(1)},$$

where r is any prime root of p, a prime of form $4m+1$, $s = r^{3m}+1/p$, $= r^{6m}+1/p$, and we may use the residues. Here $r^m = 44$, $r^{3m} = 105$, $p = 149$ (see my Solution to Question 15463). Consequently, (1) $44\,.\,105-149\,.\,31 = 1$ (see *Reprint* for a fuller discussion). Thus

$$(pq\pm 44)^2+1 = p\left[(10q\pm 3)^2+(7q\pm 2)^2\right],$$

for all values of q positive or negative, zero included. The problem now resolves itself into another: To get $(10q\pm 3)^2+(7q\pm 2)^2=\square$ by various known aids; $q = 762$.

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9862. (Capitaine De Rocquigny.)—Les n premiers nombres entiers sont dans une urne. On en tire au hasard trois. Quelle est la probabilité que l'un de ces nombres est égal à la somme des deux autres ou égal à leur demi-somme?

Solution by H. M. Taylor, M.A., F.R.S.

Let x, y, z be three numbers drawn from the urn, and let x, y, z be in descending order of magnitude.

Now let us consider the number of solutions, in unequal positive integers less than $n+1$, of

$$(1)\ x = y + z, \quad (2)\ 2y = x + z, \quad \text{and} \quad (3)\ \tfrac{1}{2}x = \tfrac{1}{2}y = z.$$

The number of solutions of each of the equations (1) and (2) is the same, and, when $n = 2p+1$, is p^2; and, when $n = 2p$, is $p(p-2)$. Both of these forms may be put into the form $\frac{1}{4}\left[(n-1)^2 - \frac{1}{2}\left\{1+(-1)^n\right\}\right]$. If we wish to find the number of selections of three numbers which will satisfy one or both of the equations (1) and (2), we must subtract from the double of the expression last given q, where $3q$ is the highest multiple of 3 less than $n+1$. The number is $\frac{1}{4}\left\{2n^2 - 4n + 1 - (-1)^n\right\} - q$. To find the chance required we have merely to divide the last result by $n!/[3!\,(n-3)!]$.

15540. (Robt. W. D. Christie.)—Solve the equation in positive integers $[\frac{1}{2}(p+e-f)][\frac{1}{2}(p+f-e)] - p[\frac{1}{2}(p-e-f)] = 1$, when p is any prime whatever of form $4m+1$. Will it be true also for n?

Solution by Lt.-Col. Allan Cunningham, R.E.

By multiplying out and rearranging, $p^2 - 2(e+f)p + \left\{(e-f)^2 + 4\right\} = 0$. Let p', p'' be the roots of this quadratic in p; then

$$p' + p'' = 2(e+f), \quad p'p'' = (e-f)^2 + 2^2 = \text{a sum of squares};$$

therefore each of p', p'' = a sum of squares; and, if p', p'' be odd, they must $= 4w+1$. Now, let p' be odd, and let it be expressed in form $p' = a^2 + (2\beta)^2$ (a odd); therefore $(a \pm kp')^2 \equiv -4\beta^2 \pmod{p'}$.

Now find k so that $(a \pm kp')/\beta$ = an (odd) integer $= \alpha$ (suppose). Then $\alpha^2 \equiv -4 \pmod{p'}$, and $\alpha^2 + 4 \equiv 0 \pmod{p'}$. Hence we may take

$$e - f = \alpha \quad \text{and} \quad p'' = \left\{(e-f)^2 + 4\right\}/p' = (\alpha^2+4)/p',$$

also $$p' + p'' = 2(e+f) = 2(\alpha + 2f);$$

therefore $$f = \tfrac{1}{2}(p' + p'' - 2\alpha), \quad e = \tfrac{1}{2}(p' + p'' + 2\alpha).$$

Thus, when $p' = 4w+1 = a^2 + (2\beta)^2$ is given, then k, α, p'', e, f are all determined as above. This shows that the condition p' = prime is unnecessary; it suffices that $p' = 4w+1 = a^2 + b^2$ (a odd, b even); or (which is the same thing) p' = product of primes of form $(4w+1)$.

An example will make this clearer. Take $p' = 221$. Here

$$p' = 13 \cdot 17 = 11^2 + 10^2, \quad \text{and} \quad 11^2 \equiv -10^2 \pmod{p'};$$

therefore $(11-221)^2 = (-210)^2 \equiv -10^2$ and $42^2 \equiv -4 \pmod{p'}$;

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therefore also $(221-42)^2 = 179^2 \equiv -4 \pmod{p'}$;
therefore $e-f = a = 179$, $p'' = (a^2+4)/p' = 145 = 5 \cdot 29$;
whence $e = 181$, $f = 2$.

Note that $p' = 13 \cdot 17$, $p'' = 5 \cdot 29$ are both composite.

The PROPOSER gives the following general solution of the Question :—

$$(ac+bd)(a^2+b^2-ac-bd)-(a^2+b^2)(ac+bd-c^2-d^2) = 1$$

(a, b arbitrary), c, d obtained from $ad-bc = \pm 1$; for, developing, we get

$$(a^2+b^2)(ac+bd)-(ac+bd)^2-(a^2+b^2)(ac+bd)+(a^2+b^2)(c^2+d^2) = (ad-bc)^2,$$

i.e.,
$$(a^2+b^2)(c^2+d^2) = (ac \pm bd)^2+(ad \mp bc)^2,$$

by Euler's theorem. Here $p = a^2+b^2$, $e = c^2+d^2$, $f = c_1^2+d_1^2$.

Ex.: $p = 109$.—Here $a = 10$, $b = 3$; $c = 3$, $d = 1$; $c_1 = 7$, $d_1 = 2$, $e = 10$, $f = 53$; whence $33 \cdot 76-109 \cdot 23 = 1$.

For a fuller discussion and the connexion with the roots see paper in *Reprint*.

10772. (G. H. BRYAN, M.A.)—A liquid film is supported on a framework in the form of an equilateral triangle of side a. If the difference of pressures (p) on the two sides of the film is so small that the surface of the film is nearly plane, prove that its equation may be written $z = \frac{1}{4}a^2 (p/T)\, \xi\eta\zeta$, where z is the perpendicular distance of any point from the plane of the triangle, ξ, η, ζ the *areal* co-ordinates of the foot of this perpendicular, and T the surface tension.

Solution by Professor A. ANDERSON, M.A., LL.D.

It is easy to see that, since ξ, η, ζ vanish at the sides of the triangle, the equation of the surface of the film when z is small is of the form $z = m\xi\eta\zeta$, where m is a small linear magnitude. It is further evident, by symmetry, that, at the centre of the triangle, the surface is spherical. Hence $p/T = 2/\rho$, where ρ is the radius of this surface.

Let O be the centre of the triangle, and P a point near O on the perpendicular from one of the vertices, and let $OP = s$. Then, for O, $z = \frac{1}{27}m$; and, for P, since $\xi = \frac{1}{3}-2s/a\sqrt{3}$, $\eta = \frac{1}{3}+s/a\sqrt{3}$, $\zeta = \frac{1}{3}+s/a\sqrt{3}$, the value of z is $m(\frac{1}{27}-s^2/3a^2) = z'$, say; therefore $z-z' = ms^2/3a^2$. But $2\rho(z-z') = s^2$; therefore $2/\rho = 4m/3a^2$; and therefore $p/T = 4m/3a^2$. Hence the equation of the surface is $z = \frac{1}{4}a^2 (p/T)\, \xi\eta\zeta$.

15503. (ROBT. W. D. CHRISTIE.)—Prove that the so-called Pellian equation may be reversed by the *dual principle* as follows:—Since

$$(4a^3+6a^2+6a+2)^2-(4a^2+4a+5)(2a^2+2a+1)^2 = -1$$

for all values of the letter, then

$$(2a^3+6a^2+6a+4)^2-(5a^2+4a+4)(a^2+2a+2)^2 = -a^6.$$

Give a few other examples, and show how to introduce the prime roots.

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Solutions (I.) *by* Professor E. B. Escott ; (II.) *by the* Proposer.

(I.) In Degen's *Canon Pellianus* are given twelve forms of numbers N for which the expression of $\sqrt{N}$ as a continued fraction may be written down at once. E. Lucas in *Bourget's Journal de Mathématiques Spéciales*, 1887, pp. 1–6, has extended the list. In *Mathesis*, 1897, p. 11, are thirty-five formulæ of this kind, and in *Crelle's Journal für reine und angewandte Mathematik*, Vol. XI. (1834), p. 332, is a list of forty-two formulæ of this kind, and the list might be indefinitely extended. Take one of these forms, for example, $N = 9a^2 + 8a + 2$. Expressed as a continued fraction, we have $\sqrt{N} = (3a+1\,;\ \dot{2},\ 1,\ 3a,\ 1,\ 2,\ \dot{6a+2})$. The convergent for the last term but one of the period is $(81a^2 + 72a + 17)/(27a + 12)$. From this we have $(81a^2 + 72a + 17)^2 - (9a^2 + 8a + 2)(27a + 12)^2 = 1$.

In a similar manner are obtained

$$(81a^2 + 90a + 26)^2 - (9a^2 + 10a + 3)(27a + 15)^2 = 1,$$

$$(25a - 7)^2 - (25a^2 - 14a + 2)\,5^2 = -1,$$

$$[2a(b+1)^2 - 1]^2 - [a^2(b+1)^2 - a](2b+2)^2 = 1,$$

all of which possess the property mentioned.

(II.) (1) Let $a = 1/a$, and multiply both sides by the square of highest powers.

(2) Some useful cases are :—

(*a*) $p = a^2 + 1^2$: roots $= \pm r^m = \pm a$, where r is any root of $p = 4m + 1$ $(4a^3 + 3a)^2 - (a^2 + 1)(4a^2 + 1)^2 = -1$, and, reversing,

$$(3a^2 + 4)^2 - (1 + a^2)(a^2 + 4)^2 = -a^6.$$

(*b*) $p = \beta^2 + 2^2$: roots $\pm r^m = \pm(2a^2 + a + 2)$;

$$\{2(a \pm 1)^3 + 2a^3\}^2 - \{(2a+1)^2 + 2^2\}\{(a \pm 1)^2 + a^2\}^2 = -1,$$

and, the reverse,

$$\{2(a+1)^3 + 2\}^2 - [\{2(a+1)\}^2 + a^2]\{(a+1)^2 + 1\}^2 = -a^6.$$

(*c*) $p = \gamma^2 + 4^2$: roots $\pm r^m = \pm(4a^2 - a + 4)$;

$$\{20a^2 - 9a + 21\}^2 - (16a^2 - 8a + 17)(25a^2 - 10a + 26)^2 = -1,$$

and, the reverse,

$$\{21a^2 - 9a + 20\}^2 - (17a^2 - 8a + 16)(26a^2 - 10a + 25)^2 = -a^4.$$

(*d*) $p = \delta^2 + 10^2$, where δ takes the four forms $10k \pm 1$, $10k \pm 3$. The corresponding roots are

$$r^m = \pm(10k^2 \pm k + 10) = \pm(30k^2 \pm 19k + 33).$$

N.B.—If $\delta = +1$, then consult form (*a*) *supra*.

(*e*) The general solution is :—

$$\{k(4n^2a^2 \mp 4na + 4n^2 + 1) + (2na \mp a + 2n)\}^2 + 1$$
$$= \{(2na \mp 1)^2 + (2n)^2\}\{(2kn+1)^2 + (2kna \mp k + a)^2\}.$$

Here $p = (2n)^2 + (2na \mp 1)^2$ (all letters arbitrary), and the roots

$$r^m = \pm(2na^2 \mp a + 2n).$$

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ERRATA.

Page 74, line 14, in first equation, *for* "$b\,(a^2/b)$" *read* "$b/(a^2/b)$."
,, 94, ,, 6, *read* "b^2e^{-2}."

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